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ON THE REPRESENTATION OF ANALYTIC FUNCTIONS BY INFINITE SERIES

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CONTENTS

Given a set of functions $\{p_k(z)\}$, necessary and sufficient conditions are known under which the basic series $\sum_{k=0}^{\infty} \Pi_k f(0)\, p_k(z)$ will represent all functions $f(z)$ in certain classes. The various cases are included in a general theory given in part II. Questions of uniqueness are discussed, and an attempt is made to initiate a theory of representation by series of the form $\sum_{k=0}^{\infty} \alpha_k p_k(z)$ which are not necessarily basic. Topological methods are used, and part I is devoted largely to preliminaries. In part III is discussed the relationship between given sets and various associated sets such as the inverse and product sets.

Introduction

A considerable theory (Whittaker 1949) has been developed of the formal series

$$\sum_{k=0}^{\infty} \Pi_k f(0)\, p_k(z) \tag{1}$$

associated with a set of given functions (frequently polynomials)

$$p_0(z), \quad p_1(z), \quad p_2(z), \quad \ldots,$$

and for certain classes E a necessary and sufficient condition has been found that all functions in E should be represented by their basic series (1). The first such problem to be solved was that in which E was the class of all functions regular in $|z| \leqslant R$, the $\{p_k(z)\}$ were polynomials satisfying a certain condition (Cannon's condition) on their degree, and *represent* meant *converge uniformly in* $|z| \leqslant R$. Various other cases have been treated, of which an account will be found in Whittaker's book. Analogies between the various cases thus treated suggests the possibility of including them all in one general theory; this has been done, and is given in part II.

The present theory not only includes the previous results as special cases, but applies also to cases not previously considered. Necessary and sufficient conditions of a different kind from those of Whittaker are also obtained. The generalization sometimes improves

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even on existing theory; for example, the theory of Cannon series is shown to generalize to the theory of absolute effectiveness for general basic series. The theory could also be applied to other situations than those considered in this paper; for example, applied to sequential spaces, it would yield results on systems of linear algebraic equations in an infinity of unknowns.

In each of the cases we wish to cover, E is a vector space over the field of complex numbers, and the mode of representation required gives a concept of convergence in E (Alexandroff & Hopf 1935, p. 27). We are led to seek a topology $\mathscr{T}$ on E giving rise to this convergence. In many cases this concept is expressible in terms of a family of semi-norms on E, the corresponding topology being the upper bound (in the lattice of all topologies on E) of the family of topologies induced by the semi-norms (Bourbaki 1940, chap. I; 1948, chap. IX); this topology is locally convex (von Neumann 1935). In our case the topology can be defined by an enumerable set of semi-norms, so that E satisfies the first axiom of countability (Hausdorff 1927, p. 229) and can therefore be metrized with an $\mathscr{F}$-metric (Birkhoff 1936; Wehausen 1938). Although we apply our results here only to vector spaces over the complex field, there is no further difficulty in assuming a general field with a non-trivial valuation.

In the other cases we wish to cover, the concept of convergence is in a quite significant sense complementary to the preceding one. We again have a family of semi-norms (they are actually norms) defined on subspaces of a vector space F, and the corresponding topology is a *lower* bound to the family of topologies induced by the norms. Both kinds of convergence are included in what we call an $\mathscr{F}_+$-*space*, but we do not embark here on a detailed investigation of such a space, contenting ourselves with such properties as are needed for the study of series representation.

In addition to the problem of finding conditions under which each $f \in E$ will be represented by the series (1), there arises the question as to whether the representations are unique. On Whittaker's definition of a 'base of polynomials', this question is answered in the affirmative for open circles and at the origin (1949, T_{41} and T_{42}). The corresponding question for closed circles was not discussed, but it will be shown here that the answer is in the negative. When the $\{p_k(z)\}$ are not polynomials, the definition of 'base' adopted by Nassif (Whittaker 1949, p. 77) does not appear to be adapted to a generalization of these results; we therefore introduce an alternative definition which does lead to such a generalization.

To say that every $f \in E$ is represented by the series (1) is by no means the same thing as to say that all such functions can be represented by some series of the form

$$\sum_{k=0}^{\infty} \alpha_k p_k(z). \tag{2}$$

It will be seen, for example, that every function regular at the origin can be represented by a series of the form

$$\alpha_0 + \alpha_1(z-1) + \alpha_2(z^2-z) + \dots, \tag{3}$$

but that the formal series (1) associated with the functions $1, (z-1), (z^2-z), \dots$ does not represent $(1-z)^{-1}$ anywhere. There is therefore a need for a theory of representation by series of the form (2), and in an attempt to initiate this, we show that if the representations (2) are unique, they coincide with the representations (1). This is not a contradiction to what has been said with regard to the series (3), for the representations are not there unique.

This question arose from a paper of Boas (1948) and was the starting point of this investigation (see Newns 1951).

Given a basic set $\{p_k(z)\}$, we can in many ways associate with it another basic set: for example, the inverse, product, derived and integrated sets have been discussed by a number of authors. In part III we shall discuss special topics of this nature, adding the transpose, transposed inverse and generalized Laurent's series.

PART I. TOPOLOGICAL PRELIMINARIES

1. On $\mathscr{F}$-spaces

Let E be an $\mathscr{F}$-space over the field K of complex numbers. The topology $\mathscr{T}_\omega$ of E is the upper bound of an increasing sequence $\{\mathscr{T}_s: s = 1, 2, \ldots\}$ of topologies defined by semi-norms $|x|_s$. The Hausdorff separation of E is expressed by the property

$$\textit{If } x \neq 0, \textit{ then for some } s, \ |x|_s \neq 0, \tag{1·1}$$

and there is no loss of generality in assuming

$$s \leqslant \sigma \quad \textit{implies} \quad |x|_s \leqslant |x|_\sigma. \tag{1·2}$$

A metric for E is given in terms of the semi-norms by

$$d(x,y) = \sum_{s=1}^{\infty} \frac{1}{2^s} \frac{|x-y|_s}{1+|x-y|_s}.$$

We shall denote by E_s^* the *dual* of E for the topology $\mathscr{T}_s$, i.e. the vector space of linear functionals defined on E and continuous in the topology $\mathscr{T}_s$.

A series $\sum_{k=0}^{\infty} \alpha_k x_k$ will be said to *represent x in $\mathscr{T}$* if it converges in this topology to the point x. A sequence $\{z_n\}$ is called a *base* for E if every $x \in E$ is represented in $\mathscr{T}_\omega$ by exactly one series $\sum_{n=0}^{\infty} \mathrm{Z}_n(x)\, z_n$, where $\mathrm{Z}_n(x) \in K$. The following is a generalization of a result of Banach (1932, p. 111; cf. also Iyer 1950, th. 7).

THEOREM 1·1. *If every $x \in E$ is represented uniquely in some $\mathscr{F}$-topology $\mathscr{T}^{(\omega)}$, coarser than $\mathscr{T}_\omega$, by a series* $\sum_{k=0}^{\infty} \Xi_k(x)\, x_k$ (where $\{x_k\}$ is a given sequence of points in the completion of E for the topology $\mathscr{T}^{(\omega)}$), *then Ξ_k is a continuous linear functional on E. If $\{x_k\} \subset E$, the set $\{\Xi_k\}$ is orthonormal to the $\{x_k\}$.*

Proof. The linearity and orthonormality of the $\{\Xi_k\}$ are immediate from the uniqueness. For each semi-norm $|x|^{(s)}$ defining $\mathscr{T}^{(\omega)}$ consider the quantity $\max_{\mu,\nu} \left| \sum_{k=\mu}^{\nu} \Xi_k(x)\, x_k \right|^{(s)}$; this is finite by the hypothesis of convergence, and we have seen that $\Xi_k(x)$ is linear, so that this quantity is a semi-norm on E. Let $\tau^{(s)}$ be the corresponding topology and $\tau^{(\omega)}$ the upper bound of the $\{\tau^{(s)}\}$; let $\mathscr{T}$ be the upper bound of the two topologies $\mathscr{T}_\omega$ and $\tau^{(\omega)}$. The topology $\mathscr{T}$ is an $\mathscr{F}$-topology on E, finer than $\mathscr{T}_\omega$. If $\{y_n\}$ is a Cauchy sequence for $\mathscr{T}$, then it is a Cauchy sequence for $\mathscr{T}_\omega$, and hence converges in $\mathscr{T}_\omega$ to some point $y \in E$. Also $\{y_n\}$ is a Cauchy sequence in $\tau^{(\omega)}$ and hence in $\tau^{(s)}$ for each s. Thus given $\epsilon > 0$, we can find $n_0(\epsilon, s)$ such that $\max_{\mu,\nu}\left|\sum_{k=\mu}^{\nu} \Xi_k(y_n - y_m)\, x_k\right|^{(s)} \leqslant \epsilon$ for $n, m \geqslant n_0$. In particular, $|\Xi_k(y_n - y_m)\, x_k|^{(s)} \leqslant \epsilon$ for $n, m \geqslant n_0$ and all k, so that by (1·1), $\{\Xi_k(y_n)\}$ is a Cauchy sequence in K, converging to ξ_k, say. Letting $m \to \infty$ above, we obtain $\max_{\mu,\nu}\left|\sum_{k=\mu}^{\nu} (\Xi_k(y_n) - \xi_k)\, x_k\right|^{(s)} \leqslant \epsilon$ for $n \geqslant n_0$. This shows in particular that $\sum_{k=0}^{\infty} \xi_k x_k$ converges in $\mathscr{T}^{(\omega)}$ to an element, y' say, in the completion of E for the topology $\mathscr{T}^{(\omega)}$, and that $y_n \to y'(\mathscr{T}^{(\omega)})$. But $y_n \to y(\mathscr{T}^{(\omega)})$, since $\mathscr{T}_\omega$ is finer than $\mathscr{T}^{(\omega)}$; hence $y = y'$ and by the uniqueness $\Xi_k(y) = \xi_k$. The foregoing now gives $y_n \to y(\tau^{(s)})$, and this being true for all s, $y_n \to y$ in $\tau^{(\omega)}$ and hence in $\mathscr{T}$; thus E is complete under $\mathscr{T}$. It follows that the two topologies $\mathscr{T}$ and $\mathscr{T}_\omega$ are identical. Hence $y_n \to y(\mathscr{T}_\omega)$ implies $y_n \to y(\mathscr{T})$ and this (as we have proved incidentally) implies $\Xi_k(y_n) \to \Xi_k(y)$, thus proving the continuity. In particular, taking $\mathscr{T}^{(\omega)} = \mathscr{T}_\omega$, we obtain:

Corollary. If $\{z_n\}$ is a base for E then $Z_n \in E^*_\omega$ and $\{Z_n\}$ is orthonormal to the $\{z_n\}$.

A series $\sum_{k=0}^{\infty} \alpha_k x_k$ will be said to *represent x absolutely* in a semi-norm topology $\mathscr{T}$ if it represents x in $\mathscr{T}$ and $\sum_{k=0}^{\infty} |\alpha_k|\,|x_k|$ converges. Such a series will be said to represent x absolutely in an $\mathscr{F}$-topology $\mathscr{T}^{(\omega)}$ if it represents x absolutely in $\mathscr{T}^{(s)}$ for all s. A base will be called an *absolute base* if its representations are absolute. Correspondingly, we have:

THEOREM 1·2. *If every $x \in E$ is represented absolutely and uniquely in some $\mathscr{F}$-topology $\mathscr{T}^{(\omega)}$, coarser than $\mathscr{T}_\omega$, by a series $\sum_{k=0}^{\infty} \Xi_k(x)\, x_k$, then the topology induced by the semi-norm $\sum_{k=0}^{\infty} |\Xi_k(x)|\,|x_k|^{(s)}$ is, for each s, coarser than $\mathscr{T}_\omega$.*

Proof. In the proof of the preceding theorem, replace

$$\max_{\mu,\nu}\left|\sum_{k=\mu}^{\nu} \Xi_k(x)\, x_k\right|^{(s)} \quad \text{by} \quad \sum_{k=0}^{\infty} |\Xi_k(x)|\,|x_k|^{(s)}$$

and the corresponding topologies $\tau^{(s)}$, $\tau^{(\omega)}$, $\mathscr{T}$ by $\bar{\tau}^{(s)}$, $\bar{\tau}^{(\omega)}$, $\bar{\mathscr{T}}$. We may now prove with little modification that $\bar{\mathscr{T}} = \mathscr{T}_\omega$, from which the result follows immediately. In particular, taking $\mathscr{T}^{(\omega)} = \mathscr{T}_\omega$:

Corollary 1. If E has an absolute base $\{z_n\}$, then its topology $\mathscr{T}_\omega$ is definable in terms of the semi-norms $N_s(x) = \sum_{n=0}^{\infty} |Z_n(x)|\,|z_n|_s$.

For $N_s(x) \geqslant |x|_s$, so that $\mathscr{T}_s$ is coarser than $\bar{\tau}_s$ for all s, and $\mathscr{T}_\omega$ is consequently coarser than $\bar{\tau}_\omega$; it is also finer than $\bar{\tau}_\omega$ by the theorem, so that the two are identical.

Corollary 2. With $N_s(x)$ as in corollary 1, given s, there exists σ such that $N_s(x) \leqslant M\,|x|_\sigma$.

Since $N_s(z_n) = |z_n|_s$, we have $N_s(x) = \sum_{n=0}^{\infty} |Z_n(x)|\, N_s(z_n)$, i.e. *the semi-norms can be so chosen that $|x|_s = \sum_{n=0}^{\infty} |Z_n(x)|\,|z_n|_s$.* With this choice, if $|z_n|_s \neq 0$, then

$$|Z_n(x)| = |z_n|_s^{-1}\, |Z_n(x)|\,|z_n|_s \leqslant |z_n|_s^{-1}\, |x|_s.$$

Hence $Z_n \in E^*_s$ and $|Z_n|_s \leqslant |z_n|_s^{-1}$. But $|Z_n(z_n)| = 1$, so that

$$|Z_n|_s\, |z_n|_s = 1 \quad \textit{whenever} \quad |z_n|_s \neq 0. \tag{1·3}$$

THEOREM 1·3. *If E has an absolute base, then every $\Xi \in E_s^*$ is of the form* $\Xi(x) = \sum_{n=0}^{\infty} \alpha_n Z_n(x)$, *where $\alpha_n \in K$, $\alpha_n = 0$ if $|z_n|_s = 0$, and otherwise $\{|\alpha_n|/|z_n|_s\}$ is bounded; and conversely.*

Note. $\mathscr{T}_s$ here refers to the semi-norms $N_s(x)$ of the previous theorem.

Proof. If $\Xi \in E_s^*$ then $\Xi(x) = \sum_{n=0}^{\infty} Z_n(x)\,\Xi(z_n)$. Write $\Xi(z_n) = \alpha_n$. Then $|\Xi(x)| \leqslant M|x|_s$ and in particular $|\alpha_n| = |\Xi(z_n)| \leqslant M|z_n|_s$. Conversely, if $|\alpha_n| \leqslant M|z_n|_s$, define $\Xi(x) = \sum_{n=0}^{\infty} \alpha_n Z_n(x)$. This converges since

$$\sum_{n=0}^{\infty} |\alpha_n|\,|Z_n(x)| \leqslant M \sum_{n=0}^{\infty} |Z_n(x)|\,|z_n|_s.$$

Hence $|\Xi(x)| \leqslant M|x|_s$ and $\Xi \in E_s^*$.

An indication of the strength of the hypothesis 'E has an absolute base' is given by:

THEOREM 1·4. *If E has an absolute base, then strong and weak sequential convergence are equivalent in E.*

Proof. Let $\{z_n\}$ be the base. Then we can use the semi-norms $|x|_s = \sum_{k=0}^{\infty} |Z_k(x)|\,|z_k|_s$. For fixed s, consider the mapping

$$u_s \;:\; x \to \{Z_k(x)\,|z_k|_s\}.$$

This is linear, maps E into the space l of absolutely convergent series and $|x|_s = \|u_s(x)\|$. If $f \in l^*$, then $\Xi(x) = f(u_s(x)) \in E_s^* \subset E_\omega^*$. Suppose $\{x_n\}$ converges weakly to zero in E. Then by definition $f(u_s(x_n)) = \Xi(x_n) \to 0$ as $n \to \infty$. This being true for all $f \in l^*$, the sequence $\{u_s(x_n)\}$ converges weakly to zero in l. By Banach (1932, p. 137), $\{u_s(x_n)\}$ converges strongly to zero in l, i.e. $\|u_s(x_n)\| = |x_n|_s \to 0$ as $n \to \infty$. This being true for each s, $x_n \to 0$ strongly in E.

2. On $\mathscr{F}_+$-spaces

Suppose we are given a vector space F over the field K and a sequence $\{F_s\}$ of subspaces of F with the properties:

$$F = \bigcup_{s=1}^{\infty} F_s, \tag{2·1}$$

$$F_s \subset F_{s+1}, \tag{2·2}$$

F_s is an $\mathscr{F}$-space under a topology $\mathscr{T}_s^{(\omega)}$, the topology induced on F_s by $\mathscr{T}_{s+1}^{(\omega)}$ being coarser than $\mathscr{T}_s^{(\omega)}$. (2·3)

A natural concept of convergence for F is that in which $x_n \to x$ is to mean $\{x_n\} \subset F_s$ *for some s and $x_n \to x(\mathscr{T}_s^{(\omega)})$*. A topology $\mathscr{T}_+$ producing this kind of convergence will be called an $\mathscr{F}_+$-topology and the space F, endowed with this topology, an $\mathscr{F}_+$-space.

As already indicated in the introduction, we are not here concerned with such questions as to whether such a topology always exists, or with any properties of such a space except those relevant to the problem in hand.

As examples of $\mathscr{F}_+$-spaces suppose the sequence $\{F_s\}$ stationary, so that $F = F_\sigma$ for some σ. Then for $s > \sigma$, $\mathscr{T}_s^{(\omega)} = \mathscr{T}_\sigma^{(\omega)}$ and F is an *$\mathscr{F}$-space* under $\mathscr{T}_+ = \mathscr{T}_\sigma^{(\omega)}$. If, on the other hand, the sequence $\{F_s\}$ is strictly increasing, whilst the topology induced on F_s by $\mathscr{T}_{s+1}^{(\omega)}$ is identical with $\mathscr{T}_s^{(\omega)}$, we can topologize F as an *$\mathscr{L}\mathscr{F}$-space*. In either case F is an $\mathscr{F}_+$-space, the first trivially, the second by Dieudonné & Schwartz (1950, p. 70).

Our chief application is to duals of certain $\mathscr{F}$-spaces. Let E be an $\mathscr{F}$-space with an absolute base $\{z_n\}$ such that, for each s,

$$\sum_{n=0}^{\infty} |z_n|_s / |z_n|_{s+1} < \infty. \tag{2·4}$$

By (1·3) this is equivalent to

$$\sum_{n=0}^{\infty} |z_n|_s \, |Z_n|_{s+1} < \infty. \tag{2·5}$$

In the vector space $F = E^*_\omega$ take $F_s = E^*_s$; then it is known that (2·1) holds. Moreover, K being complete, E^*_s is a *Banach space* under the topology $\mathscr{T}_s{}^*$ induced by the norm $|\Xi|_s$, and since by (1·2) $|\Xi(x)| \leqslant |\Xi|_s |x|_s \leqslant |\Xi|_s |x|_{s+1}$, we see that $|\Xi|_{s+1} \leqslant |\Xi|_s$ for all $\Xi \in E^*_s$; this shows that (2·2) and (2·3) hold.

Let $\mathscr{T}_+$ be any topology (compatible, of course, with the algebraic structure) on E^*_ω, finer than the weak topology but inducing on each E^*_s a topology coarser than $\mathscr{T}_s{}^*$. By theorem 1·3, if $\Xi \in E^*_\omega$, the series $\sum_{n=0}^{\infty} \Xi(z_n) Z_n$ represents Ξ weakly in E^*_ω. But if $\Xi \in E^*_s$,

$$\sum_{n=0}^{\infty} |\Xi(z_n)| \, |Z_n|_{s+1} \leqslant |\Xi|_s \sum_{n=0}^{\infty} |z_n|_s \, |Z_n|_{s+1} < \infty$$

by (2·5), showing that *the series represents* Ξ *strongly and absolutely in* E^*_{s+1}. These representations are, moreover, *unique*, since if $\sum_{k=0}^{\infty} \alpha_k Z_k = 0(\mathscr{T}_+)$, $\sum_{k=0}^{\infty} \alpha_k Z_k(z_n) = \alpha_n = 0$ for each n.

Now let $\{\Xi_n\}$ be a Cauchy sequence in E^*_ω. Then $\{\Xi_n\}$ is a weak Cauchy sequence, hence weakly bounded, hence equi-continuous (Dieudonné & Schwartz 1950, p. 62, § 2 and p. 73, th. 2). This means that for some s, $\Xi_n \in E^*_s$ for all n and $\{\Xi_n\}$ is bounded for $\mathscr{T}_s{}^*$; let M be a bound for $\{|\Xi_n|_s\}$. Then

$$\begin{aligned} |\Xi_n - \Xi_m|_{s+1} &\leqslant \sum_{k=0}^{\infty} |\Xi_n(z_k) - \Xi_m(z_k)| \, |Z_k|_{s+1} \\ &\leqslant \sum_{k=0}^{\infty} (|\Xi_n(z_k)| + |\Xi_m(z_k)|) \, |Z_k|_{s+1} \\ &\leqslant 2M \sum_{k=0}^{\infty} |z_k|_s \, |Z_k|_{s+1} < \infty \quad \text{by (2·5)}. \end{aligned}$$

Since $\{\Xi_n(z_k)\}$ is, for each k, a Cauchy sequence in K, and we have uniform convergence in n and m, we see that $\{\Xi_n\}$ is a Cauchy sequence in E^*_{s+1}, convergent since this space is complete.

Recall that a *base* for an $\mathscr{F}_+$-space F will be a set $\{z_n\}$ of points of F such that every $x \in F$ is represented uniquely in the topology $\mathscr{T}_+$ on F by a series $\sum_{n=0}^{\infty} Z_n(x) z_n$. By the definition of $\mathscr{T}_+$, each such series must converge in some $\mathscr{T}_s^{(\omega)}$; if this convergence is absolute, we shall call $\{z_n\}$ an *absolute base* for F. With this terminology we may summarize as follows:

Theorem 2·1. *If the* $\mathscr{F}$*-space* E *has an absolute base* $\{z_n\}$ *satisfying* (2·4), *then its dual space* E^*_ω *can be topologized as an* $\mathscr{F}_+$*-space and the orthonormal set* $\{Z_n\}$ *forms an absolute base for that space.*

We now show that the study of series representation in $\mathscr{F}_+$-spaces can be reduced to the corresponding study in $\mathscr{F}$-spaces.

Theorem 2·2. *Let* F *be an* $\mathscr{F}_+$*-space and* $\{x_k\}$ *a given sequence of points of* F. *For any fixed* s, *suppose that each* $x \in F_s$ *is represented* (not necessarily uniquely) *in the topology* $\mathscr{T}_+$ *by a series of* *the form* $\sum_{k=0}^{\infty} \alpha_k x_k$. *Then there exists* σ (depending on s but not on x) *such that each* $x \in F_s$ *is so represented in the topology* $\mathscr{T}_\sigma^{(\omega)}$.

Proof. We may assume that no x_k is zero. To each t corresponds a subset $F_s^{(t)}$ of F_s, namely, the set of all x which are represented in $\mathscr{T}_t^{(\omega)}$. By the definition of $\mathscr{T}_+$, $F_s = \bigcup_{t=s}^{\infty} F_s^{(t)}$. But F_s, being a complete metric space, is of the second category (Banach 1932, p. 14, th. 2); hence for some σ, $F_s^{(\sigma)}$ *is a set of the second category in* F_s.

Let E be the set of all sequences $y = \{\alpha_k\}$ such that $\sum_{k=0}^{\infty} \alpha_k x_k$ converges in $\mathscr{T}_\sigma^{(\omega)}$ to a point $x \in F_s$, and denote this correspondence between x and y by $x = u(y)$. Under the usual definitions of algebraic operations E becomes a vector space over K and u a linear mapping of E into F_s. The quantity $|y|^{(\iota)} = \max_{\mu,\nu} \left| \sum_{k=\mu}^{\nu} \alpha_k x_k \right|_\sigma^{(\iota)}$ is for each ι a semi-norm on E, and if $y \neq 0$ then $|y|^{(\iota)} \neq 0$ for some ι; let $d_0(y_1, y_2)$ be the metric constructed from these semi-norms (§ 1) and let $d_s(x_1, x_2)$ be the metric for F_s. Then $d(y_1, y_2) = d_0(y_1, y_2) + d_s(u(y_1), u(y_2))$ is an $\mathscr{F}$-metric for E, and we shall show that E is complete under this metric. Let $\{y_n\}$ be a Cauchy sequence in E; then $\{u(y_n)\}$ is a Cauchy sequence in F_s, and hence converges in $\mathscr{T}_s^{(\omega)}$ to a point $x \in F_s$. Also $\{y_n\}$ is a Cauchy sequence in each topology $\tau^{(\iota)}$ (induced by $|y|^{(\iota)}$), so that if $y_n = \{\alpha_k^{(n)}\}$, given $\epsilon > 0$, we can find $n_0(\epsilon, \iota)$ such that $\max_{\mu,\nu} \left| \sum_{k=\mu}^{\nu} (\alpha_k^{(n)} - \alpha_k^{(m)})\, x_k \right|_\sigma^{(\iota)} \leqslant \epsilon$ for $n, m \geqslant n_0$. By (1·1), $\{\alpha_k^{(n)}\}$ is a Cauchy sequence in K for each k, converging to α_k, say. Letting $m \to \infty$ above, $\max_{\mu,\nu} \left| \sum_{k=\mu}^{\nu} (\alpha_k^{(n)} - \alpha_k)\, x_k \right|_\sigma^{(\iota)} \leqslant \epsilon$ for $n \geqslant n_0$. This shows in particular that $\sum_{k=0}^{\infty} \alpha_k x_k$ converges in $\mathscr{T}_\sigma^{(\omega)}$ to a point x', say, and that $u(y_n) \to x' (\mathscr{T}_\sigma^{(\omega)})$. But $u(y_n) \to x (\mathscr{T}_\sigma^{(\omega)})$ by (2·3) and the above, so that $x = x'$. This shows that $y = \{\alpha_k\} \in E$ and that $u(y) = x$. The above now means that $y_n \to y(\tau^{(\iota)})$ for each ι so that $d(y_n, y) \to 0$, and E is an $\mathscr{F}$-space.

Since, by definition of d, u is continuous on E, and $u(E) = F_s^{(\sigma)}$ is of the second category in F_s, we must have $F_s^{(\sigma)} = F_s$ (Banach 1932, p. 38, th. 3). This is what we set out to prove. Similarly:

THEOREM 2·3. *Let F be an* $\mathscr{F}_+$*-space and* $\{x_k\}$ *a given sequence of points of F. For any fixed s, suppose that each* $x \in F_s$ *is represented absolutely in the topology* $\mathscr{T}_+$ *by a series of the form* $\sum_{k=0}^{\infty} \alpha_k x_k$. *Then there exists* σ (depending on s but not on x) *such that each* $x \in F_s$ *is so represented in the topology* $\mathscr{T}_\sigma^{(\omega)}$.

Proof. In the preceding proof replace 'represented' by 'represented absolutely' in the definitions of $F_s^{(\sigma)}$ and E; replace the semi-norms $\max_{\mu,\nu} \left| \sum_{k=\mu}^{\nu} \alpha_k x_k \right|_\sigma^{(\iota)}$ by $\sum_{k=0}^{\infty} |\alpha_k|\, |x_k|_\sigma^{(\iota)}$ giving topologies $\bar{\tau}^{(\iota)}$. The proof now needs little further modification.

If we now wish to represent, or represent absolutely, an $\mathscr{F}_+$-space F in its $\mathscr{F}_+$-topology $\mathscr{T}_+$, by series of the form $\sum_{k=0}^{\infty} \alpha_k x_k$, we have to represent each subspace F_s in this manner. We see from these two theorems that this is equivalent to representing F_s in some $\mathscr{F}$-topology $\mathscr{T}_\sigma^{(\omega)}$, i.e. that the problem reduces to that of representing $\mathscr{F}$-spaces in (coarser) $\mathscr{F}$-topologies.

Suppose that in theorem 2·2 the expansions are *unique*; then by theorem 1·1 they may be written $\sum_{k=0}^{\infty} \Xi_k(x)\, x_k$, where $\Xi_k \in F_s^*$. In particular, if $\{z_n\}$ is a base for F, $Z_n \in \bigcap_{s=1}^{\infty} F_s^*$. In the special cases of $\mathscr{F}_+$-spaces mentioned above, $\bigcap_{s=1}^{\infty} F_s^* = F^*$ (the dual of F for the topology $\mathscr{T}_+$), so that the corollary to theorem 1·1 holds for these spaces also; this corollary is true more generally but we shall not prove this here.

We conclude this section with a property of series on any linear topological space, and in particular on an $\mathscr{F}_+$-space.

THEOREM 2·4. *Let E be a linear topological space and G a proper vector subspace of E. Suppose that every $x \in \complement G$ is represented* (in the topology of E) *by a series of the form* $\sum_{k=0}^{\infty} \alpha_k x_k$. *Then every $x \in E$ is so represented.*

Proof. Given an $x \in G$ take any $y \in \complement G$. Then $x - y \in \complement G$, so that $y = \sum_{k=0}^{\infty} \beta_k x_k$ and $x - y = \sum_{k=0}^{\infty} \gamma_k x_k$. Hence $x = (x-y) + y = \sum_{k=0}^{\infty} (\beta_k + \gamma_k)\, x_k$.

Similarly we could prove:

THEOREM 2·5. *Let G be a proper subspace of the $\mathscr{F}_+$-space F and suppose that every $x \in \complement G$ is represented absolutely in $\mathscr{T}_+$ by a series of the form* $\sum_{k=0}^{\infty} \alpha_k x_k$. *Then every $x \in F$ is so represented.*

3. SPACES OF FUNCTIONS REGULAR IN SIMPLY-CONNECTED DOMAINS

Before passing on to our chief application it is convenient to list some standard notations.

A *domain* (connected open set) in the complex plane will be denoted by D, its closure by $\overline{D}$. We write $\overline{D}_-$ for some unspecified compact region contained in D, and D_+ for some unspecified domain containing $\overline{D}$. A regular closed curve is denoted by C and its interior and exterior domains by $D(C)$, $E(C)$.

If C is the circle $|z| = R$, we shall write $D(R)$ for $D(C)$, and $E(R)$ for $E(C)$. We shall also write $D(\infty)$ for the whole plane and $D_+(0)$ for some unspecified circle of positive radius.

All our functions are single-valued. By *f is regular in D* we shall mean *f is a branch of an analytic function, regular in D*. Two different branches of the same analytic function are regarded as distinct regular functions.

We denote by $H(D)$ the class of functions regular in D, and by $\overline{H}(D)$ the class of functions regular in $\overline{D}$ (i.e. in some D_+). The class $H(R)$ may be regarded as the class of all power series with radius of convergence not less than R. In particular $H(\infty)$ is the class of integral functions and $\overline{H}(0)$ the class of functions regular at the origin.

Under the usual definitions of algebraic operations, $H(D)$ becomes a vector space over the field of complex numbers. A natural concept of convergence for $H(D)$ is that of *convergence in D which is uniform in any $\overline{D}_-$*. Let A_s be an increasing sequence of compact sets, such that $\bigcup_{s=1}^{\infty} A_s = D$. Define the topology $\mathscr{T}_s$ by the semi-norms $\max_{z \in A_s} |f(z)|$. Then it is easily seen that convergence in $\mathscr{T}_\omega$ is the sort we require. The space is complete by Cauchy's general principle of convergence and Weierstrass's double-series theorem. Equally obviously, convergence in $\mathscr{T}_s$ is strictly weaker than convergence in $\mathscr{T}_\omega$, so that $H(D)$ *is a non-normable* $\mathscr{F}$*-space*.

We require a base for $H(D)$. In the case of circles, we have the set $\{z^n\}$, which is known to be an absolute base for $H(R)$. For $H(C)$, we could use the *Faber polynomials* of C, which Faber (1903) has shown to be an absolute base. In the general case, if the boundary of D has only one point say, we can use the set $\{(z-z_0)^{-n}\}$. (When the point is the point at infinity, this case, the space of all integral functions, is included in the case of circles.) Finally, for a general simply-connected domain with more than one boundary point, let $w = \psi(z)$ be a conformal transformation mapping D on to the unit circle $D(1)$. The set $\{[\psi(z)]^n\}$ forms an absolute base, since if $f(z)$ is regular in D we have $f(z) = f(\psi^{-1}(w)) = \sum_{n=0}^{\infty} a_n w^n = \sum_{n=0}^{\infty} a_n[\psi(z)]^n$ and the series converges in D, uniformly in any $\overline{D}_-$. The uniqueness is a consequence of that of Taylor series.

It follows that $H(D)$ *is separable, has an absolute base, and that strong and weak sequential convergence in* $H(D)$ *are equivalent.*

By theorem 1·3 we see that every linear functional on $H(R)$ continuous in $\mathscr{T}_s$ is of the form $\Phi(f) = \sum_{n=0}^{\infty} c_n a_n$, where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\{|c_n|\,|z^n|_s^{-1}\}$ is bounded. Thus every $\Phi \in H^*(R)$ is of the form $\Phi(f) = \sum_{n=0}^{\infty} c_n a_n$, where $\overline{\lim}_{n\to\infty} |c_n|^{1/n} < R$; i.e. $H^*(R)$ *can be identified with* $\overline{H}(1/R)$. Further, if we write $\phi(z) = \sum_{n=0}^{\infty} c_n z^{-n-1}$, we obtain:

THEOREM 3·1. *Every* $\Phi \in H^*(R)$ *is of the form* $\Phi(f) = \frac{1}{2\pi i}\int_C f(z)\,\phi(z)\,dz$, *where* $\phi(z)$ *is regular in* $E(r)$ *for some* $r<R$, $\phi(\infty) = 0$ *and* C *is any circle* $|z| = \rho$, $r<\rho<R$.

THEOREM 3·2. *The set* $(\overline{H}R)$ *is of the first category in the space* $H(R)$.

Proof. If $R<\rho<\infty$, then $H(\rho)$ *is a set of the first category in the space* $H(R)$. For the natural (inclusion) mapping of the *space* $H(\rho)$ onto the subset $H(\rho)$ of the space $H(R)$ is continuous, the topology of $H(\rho)$ being finer than that of $H(R)$. Hence $H(\rho)$ is either of the first category or identical with $H(R)$, and the latter alternative is false. Finally, $\overline{H}(R) = \bigcup_{n=1}^{\infty} H(R_n)$, where $R_n = R+1/n$.

It is instructive to consider the statement of topological theorems when interpreted in terms of $H(D)$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\rho)$, then the corollary to theorem 1·1 shows that the Taylor coefficient a_n is a continuous linear functional on $H(\rho)$. The inequality expressing this fact reduces to Cauchy's inequality. If we denote $\max_{|z|=r} |f(z)|$ by $M(r)$ and the majorant $\sum_{n=0}^{\infty} |a_n|\, r^n$ by $\mathfrak{M}(r)$, corollary 2 to theorem 1·2 gives $\mathfrak{M}(r) \leqslant AM(R)$ for any $R>r$ (since we can vary ρ) and some constant A (depending on r and R). This inequality can easily be obtained by an application of Schwarz and Cauchy's inequalities (strong form), but the corresponding inequality for, say, $H(C)$ with Faber polynomials as base, is less obvious.

4. Spaces of functions regular in closed circles

We have seen above that the vector space $\overline{H}(r)$ can be regarded as the dual of an $\mathscr{F}$-space (namely, $H(1/r)$). In accordance with § 2, $\overline{H}(r)$ can be topologized as an $\mathscr{F}_+$-space. Here, convergence in $\mathscr{T}_+$ means *uniform convergence in some* $\overline{D_+(r)}$. In connexion with representation by series, this concept has been used in the special case $r=0$, and is called *representation at the origin* (Whittaker 1949, chap. VI). For $r \neq 0$, the concept of convergence hitherto considered has been that of uniform convergence in $\overline{D(r)}$. We shall later see instances in which properties of $H(R)$ correspond to properties of $\bar{H}(r)$ with the topology $\mathscr{T}_+$, whereas $\bar{H}(r)$, regarded as a space with the topology of uniform convergence in $\overline{D(r)}$, generally fails to have that property. We shall see also that in cases where $\bar{H}(r)$ (with convergence in $\overline{D(r)}$) has been proved to have certain properties, the conditions assumed have implied that, in the case in question, *convergence in* $\overline{D(r)}$ *and in* $D_+(r)$ *are equivalent.* It is remarkable that this idea of representation in closed circles should, in this subject, have received more attention than the more natural concepts of representation in open circles and at the origin.

If we take, as base for $H(1/r)$, the Taylor base $\{z^n\}$, we see from § 3 that the biorthogonal set is again the Taylor base $\{z^n\}$ in $\bar{H}(r)$. Theorem 2·1 states the well-known fact that every $f \in \bar{H}(r)$ is represented absolutely by its Taylor series in some $\overline{D_+(r)}$. From theorem 2·2 we obtain:

THEOREM 4·1. *Let* $\{p_k(z)\}$ *be any sequence of functions in* $\bar{H}(r)$ *such that every* $f \in H(\rho)$ (*for a given* $\rho > r$) *is represented in some* $D_+(r)$ *by a series of the form* $\sum_{k=0}^{\infty} c_k p_k(z)$. *Then there exists* $R > r$ *such that every such* f *is so represented in* $\overline{D(R)}$.

5. Spaces of integral functions

Consider the class $I(\rho, \sigma)$ of all integral functions of increase not exceeding order ρ, type σ $(0<\rho<\infty)$. This class is a vector subspace of the space of all integral functions. Consider the mapping $u(f) = \sum_{n=0}^{\infty} \left(\frac{n}{e\rho}\right)^{n/\rho} a_n z^n$, where $f(z) = \sum_{n=0}^{\infty} a_n z^n$. If $f \in I(\rho, \sigma)$, then

$$\overline{\lim_{n\to\infty}} \left[\left(\frac{n}{e\rho}\right)^{1/\rho} |a_n|^{1/n}\right] \leqslant \sigma^{1/\rho},$$

so that $u(f) \in H(\sigma^{-1/\rho})$. Hence u is a $(1, 1)$ linear mapping of $I(\rho, \sigma)$ onto $H(\sigma^{-1/\rho})$.

Let us now write d_0 for the metric of $H(\sigma^{-1/\rho})$, and define $d(f, g) = d_0(u(f), u(g))$ as a metric for $I(\rho, \sigma)$. With this metric $I(\rho, \sigma)$ becomes an $\mathscr{F}$*-space*, and u an isomorphism of the $\mathscr{F}$-spaces $I(\rho, \sigma)$ and $H(\sigma^{-1/\rho})$. Now $d(f_n, f) \to 0$ as $n \to \infty$ is equivalent to $|u(f_n) - u(f)|_s \to 0$ as $n \to \infty$ for all s; i.e. to $\sum_{k=0}^{\infty} |a_k^{(n)} - a_k| \left(\frac{k}{e\rho}\right)^{k/\rho} r^k \to 0$ for all $r < \sigma^{-1/\rho}$. Thus $f_n \to f$ in the topology of $I(\rho, \sigma)$ implies $f_n \to f$ in the topology of $H(\infty)$. In other words, *the topology induced on* $I(\rho, \sigma)$ *by that of* $H(\infty)$ *is coarser than its* $\mathscr{F}$*-topology.* As in theorem 3·2, we deduce that functions of finite order form a set of the first category in the space of all integral functions (cf. Iyer 1948, th. 10)

The images of the absolute base $\{z^n\}$ for $H(\sigma^{-1/\rho})$ under u, namely, $\{(n/e\rho)^{n/\rho} z^n\}$, form an *absolute base* for $I(\rho, \sigma)$. Since these are merely multiples of the functions $\{z^n\}$, we may take, for simplicity, the set $\{z^n\}$ as an absolute base. The isomorphism of $I(\rho, \sigma)$ with $H(\sigma^{-1/\rho})$ shows that we could state analogous properties for $I(\rho, \sigma)$ to those stated above for $H(R)$.

PART II. A GENERAL THEORY OF BASIC SERIES

6. Basic series on $\mathscr{F}_+$-spaces

Let F be an $\mathscr{F}_+$-space over the field K and $\{x_k\}$ a sequence of points in F. We are concerned with the possibility of representing each $x \in F$ by a series of the form $\sum_{k=0}^{\infty} \alpha_k x_k$, convergent in the topology $\mathscr{T}_+$ of F. Suppose $\{z_n\}$ is a base for F. Then every $x \in F$ is represented uniquely by

$$x = \sum_{n=0}^{\infty} \mathrm{Z}_n(x)\, z_n \quad (\mathscr{T}_+). \tag{6·1}$$

If now every $x \in F$ is to be represented in the form

$$x = \sum_{k=0}^{\infty} \alpha_k x_k, \tag{6·2}$$

then, in particular, each z_n must be so represented. We therefore make the following definition:

A set $\{p_k: k = 0, 1, 2, \ldots\}$ of points of the $\mathscr{F}_+$-space F is called a basic set if no p_k is zero and we are given a double array $(\Pi_k(z_n))$ of elements of the field K such that

$$z_n = \sum_{k=0}^{\infty} \Pi_k(z_n)\, p_k \quad (\mathscr{T}_+). \tag{β}$$

We use the term 'basic set' to avoid confusion with the term 'base' which we have already used in its usual meaning. The hypothesis (β) is necessary if there are to be 'enough' points in the set to span the space. Since we are not here concerned with uniqueness, we have no hypothesis designed to prevent there being 'too many' points in a basic set. It could happen, for instance, that $p_i = p_j$ for some $i \neq j$; hence the same set $\{p_k\}$ may form more than one basic set (with different coefficients $\Pi_k(z_n)$). We do, however, assume that there are no elements zero in a basic set, since this enables us to express our conditions for 'effectiveness' (§7 below) more neatly.

If $\{p_k\}$ is a basic set, then by (6·1) and (β), we have

$$x = \sum_{n=0}^{\infty} \mathrm{Z}_n(x)\, z_n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \mathrm{Z}_n(x)\, \Pi_k(z_n)\, p_k,$$

so that formally, inverting the order of summation,

$$x = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \Pi_k(z_n)\, \mathrm{Z}_n(x)\, p_k = \sum_{k=0}^{\infty} \Pi_k(x)\, p_k,$$

where we have written

$$\Pi_k(x) = \sum_{n=0}^{\infty} \Pi_k(z_n)\, \mathrm{Z}_n(x). \tag{6·3}$$

There is no ambiguity in this notation since $\{z_n\}$ is a base and consequently $\mathrm{Z}_n(z_m) = \delta_m^n$. The formal series

$$B(x) = \sum_{k=0}^{\infty} \Pi_k(x)\, p_k \tag{6·4}$$

is called the *basic series* of x, and the formal coefficient $\Pi_k(x)$ a *basic coefficient*. The series (6·4) is of the required form (6·2), and there arises the corresponding question: 'under what circumstances will every $x \in F$ be represented (in $\mathscr{T}_+$) by its basic series?'

If every x in a space E is represented in a topology $\mathscr{T}$ by its basic series, we shall say that the basic set is *effective for* E *in* $\mathscr{T}$; we similarly define *absolute effectiveness*.

Given a subspace F_s of F with the properties defined in § 2, we consider effectiveness for F_s in $\mathscr{T}_+$. Now we saw in part I that $Z_n \in F_s^*$, so that $\sum_{n=0}^{N} \Pi_k(z_n) Z_n \in F_s^*$. Moreover, the limit of a sequence of continuous linear functionals on an $\mathscr{F}$-space being continuous (Banach 1932, p. 23, th. 4), we obtain:

Theorem 6·1. *If* $\Pi_k(x)$ *is defined* (i.e. the series (6·3) converges) *for all* $x \in F_s$, *then* $\Pi_k \in F_s^*$.

Theorem 6·2. *If a basic set is effective for* F_s *in* $\mathscr{T}_+$, *then it is effective for* F_s *in some* $\mathscr{T}_\sigma^{(\omega)}$.

Proof. This theorem is the analogue for basic representations of theorem 2·2, and the proof needs little modification (using theorem 6·1). The following is an alternative argument: let $F_s^{(\sigma)}$ be a subset of F_s such that every $x \in F_s^{(\sigma)}$ is represented in $\mathscr{T}_\sigma^{(\omega)}$ and $F_s^{(\sigma)}$ is of the second category in F_s. Write

$$B_n(x) = \sum_{k=0}^{n} \Pi_k(x)\, p_k. \tag{6·5}$$

Then by theorem 6·1, B_n is a continuous linear mapping of F_s into F_σ.† Hence the subspace $F_s^{(\sigma)}$, being the set of points at which this sequence converges, is Borel measurable (Banach 1932, p. 18, th. 9). Hence $F_s^{(\sigma)} = F_s$ by Banach (1932, p. 36, th. 1). Similarly, we could prove:

Theorem 6·3. *If a basic set is absolutely effective for* F_s *in* $\mathscr{T}_+$, *then it is absolutely effective for* F_s *in some* $\mathscr{T}_\sigma^{(\omega)}$.

Theorems 2·4 and 2·5 also have their analogues:

Theorem 6·4. *Let* E *be a linear topological space and* G *a proper linear subspace of* E. *If a basic set is effective for* $\complement G$ *in the topology* $\mathscr{T}$ *of* E, *then it is effective for* E *in* $\mathscr{T}$.

Theorem 6·5. *Let* G *be a proper subspace of the* $\mathscr{F}_+$*-space* F *and suppose a basic set is absolutely effective for* $\complement G$ *in* $\mathscr{T}_+$. *Then the basic set is absolutely effective for* F *in* $\mathscr{T}_+$.

These follow similarly from the linearity of B. If, in theorem 6·4, we take $\mathscr{T}$ to be the topology induced by that of $H(R)$ on $E = H(\rho)$, and G to be $\bar{H}(\rho)$, we have a theorem of Whittaker (1949, p. 37, T_{25}); Whittaker's proof (p. 15, T_6) seems to depend on a deep result (p. 32, T_{22}), while our proof is trivial.

7. Conditions for effectiveness

For the basic series of x to represent x in the topology $\mathscr{T}_+$ there are three separate requirements:

(i) For each k, the series defining $\Pi_k(x)$ must converge (in the topology of K);
(ii) The basic series $B(x)$ must converge in the topology $\mathscr{T}_+$;
(iii) The point $B(x)$ so defined must be the point x.

It may be shown by examples (Whittaker 1949, p. 63, E_{13}; p. 29, E_{10}; p. 64, E_{14}) that any of these properties may fail, even though those preceding (if any) hold. We shall show,

† If, for some k, $p_k \notin F_\sigma$, then $\Pi_k(x) = 0$ on $F_s^{(\sigma)}$ by definition of $F_s^{(\sigma)}$; hence $\Pi_k(x) = 0$ on F_s, since $F_s^{(\sigma)}$ is dense in F_s.

however, that if (ii) holds for all $x \in F$, then (iii) holds, so that (ii) and (iii) are equivalent for our purposes.

THEOREM 7·1. *If the series defining* $B(x)$ *converges* $(\mathscr{T}_+)$ *for all* $x \in F$ *then* $B(x) = x$.

Proof. In the notation (6·5), as in the proof of theorem 6·1, B_n is a continuous linear mapping of F_s into some F_σ, so that B is continuous on F_s. This being true for each s,

$$B(x) = B\left(\sum_{n=0}^{\infty} \mathrm{Z}_n(x)\, z_n\right) = \sum_{n=0}^{\infty} \mathrm{Z}_n(x)\, B(z_n) = \sum_{n=0}^{\infty} \mathrm{Z}_n(x)\, z_n = x,$$

since the hypothesis (β) may be written $B(z_n) = z_n(\mathscr{T}_+)$.

The topology $\mathscr{T}_s^{(\omega)}$ of F_s is the upper bound of the topologies $\mathscr{T}_s^{(k)}$ induced by the seminorms $|x|_s^{(k)}$ defined on F_s. Consider the expression

$$q_\sigma^{(\iota)}(x) = \max_{\mu,\,\nu} \left| \sum_{k=\mu}^{\nu} \Pi_k(x)\, p_k \right|_\sigma^{(\iota)}. \tag{7·1}$$

This will, of course, be defined only if $\Pi_k(x)$ is defined for all k. Suppose it is defined (and finite) for all $x \in F_s$ and all ι: then since Π_k is linear, $q_\sigma^{(\iota)}(x)$ is a semi-norm on F_s inducing a topology $\tau_\sigma^{(\iota)}$, and these topologies will have an upper bound $\tau_\sigma^{(\omega)}$. In this terminology we give our general condition for effectiveness.

THEOREM 7·2. *For the basic set* $\{p_k\}$ *to be effective for* F *in* $\mathscr{T}_+$ *it is necessary and sufficient that given* s, *there exists* σ *such that* $\tau_\sigma^{(\omega)}$ *is defined on* F_s *and is coarser than* $\mathscr{T}_s^{(\omega)}$; *or equivalently that for each* s *we can find* σ *such that, given* ι, *there exist* M, j *such that* $q_\sigma^{(\iota)}(x) \leqslant M\,|x|_s^{(j)}$ *for all* $x \in F_s$.

Proof. The equivalence of the two conditions is immediate. If $\{p_k\}$ is effective for F in $\mathscr{T}_+$, then by theorem 6·2, given s, $\{p_k\}$ is effective for F_s in some $\mathscr{T}_\sigma^{(\omega)}$. By theorem 6·1, for each ι, $q_\sigma^{(\iota)}(x)$ is lower semi-continuous, and the necessity is a consequence of Bourbaki (1948, chap. IX, §5, th. 2) since $q_\sigma^{(\iota)}(x)$ is a semi-norm. For the sufficiency, given s, let $\mathscr{T}_t^{(\omega)}$ be a topology in which F_s is represented by the base $\{z_n\}$ (the existence of which was proved in theorem 2·2). By hypothesis, we can find σ such that for each ι there exist M, j such that $q_\sigma^{(\iota)}(x) \leqslant M\,|x|_t^{(j)}$ for all $x \in F_t$, and *a fortiori* for all $x \in F_s$. Since $|B_n(x)|_\sigma^{(\iota)} \leqslant q_\sigma^{(\iota)}(x)$, the sufficiency follows from Bourbaki (1949, chap. X, §3, prop. 4) by taking 'E' to be F_s with topology $\mathscr{T}_t^{(\omega)}$, and 'F' to be F_σ with its topology $\mathscr{T}_\sigma^{(\omega)}$; the set of linear combinations of the $\{z_n\}$ is, of course, everywhere dense.

In the foregoing argument there is a slight complication owing to the fact that $\{z_n\}$ is not necessarily a base for F_s (with its usual topology $\mathscr{T}_s^{(\omega)}$). It is this fact that prevents a complete reduction from $\mathscr{F}_+$-spaces and topologies to $\mathscr{F}$-spaces and topologies (cf. §2). From the proof, however, we can state:

Corollary. Let E *be an* $\mathscr{F}$-*space with a base* $\{z_n\}$ *and* $\mathscr{T}^{(\omega)}$ *an* $\mathscr{F}$-*topology on* E, *coarser than the topology* $\mathscr{T}_\omega$ *of* E. *Then for the basic set* $\{p_k\}$ *to be effective for* E *in* $\mathscr{T}^{(\omega)}$ *it is necessary and sufficient that* $q^{(\sigma)}(x)$ *be defined on* E *for all* σ *and that* $\tau^{(\omega)}$ *be coarser than* $\mathscr{T}_\omega$; *or, equivalently, that for each* σ *there exist* M, s *such that* $q^{(\sigma)}(x) \leqslant M\,|x|_s$ *for all* $x \in E$.

For a space with an absolute base, the condition simplifies.

THEOREM 7·3. *If* $\{z_n\}$ *is an absolute base for* F, *then for* $\{p_k\}$ *to be effective for* F *in* $\mathscr{T}_+$ *it is sufficient* (also necessary) *that for each* s *we can find* σ *such that, given* ι, *there exist* M, j *such that* $q_\sigma^{(\iota)}(z_n) \leqslant M\,|z_n|_s^{(j)}$ *for all* n.

Proof. Let $\mathscr{T}_t^{(\omega)}$ be a topology in which F_s is represented absolutely by the base $\{z_n\}$ (the existence of which was proved in theorem 2·3). Then by hypothesis we can find σ such that for each ι there exist M, l such that $q_\sigma^{(\iota)}(z_n) \leqslant M \mid z_n \mid_t^{(l)}$ for all n. Given k, choose ι so that $\mid p_k \mid_\sigma^{(\iota)} \neq 0$. Then for any $x \in F_s$,

$$\sum_{n=0}^{\infty} \mid \Pi_k(z_n)\, Z_n(x) \mid \leqslant [\mid p_k \mid_\sigma^{(\iota)}]^{-1} \sum_{n=0}^{\infty} \mid Z_n(x) \mid q_\sigma^{(\iota)}(z_n) \leqslant M' \sum_{n=0}^{\infty} \mid Z_n(x) \mid \mid z_n \mid_t^{(l)} < \infty$$

by definition of t. From (6·3) we see that Π_k and hence B_n is defined on F_s and

$$q_\sigma^{(\iota)}(x) \leqslant 2 \max_n \mid B_n(x) \mid_\sigma^{(\iota)} = 2 \max_n \mid \sum_{k=0}^{\infty} Z_k(x)\, B_n(z_k) \mid_\sigma^{(\iota)}$$

$$\leqslant 2 \sum_{k=0}^{\infty} \mid Z_k(x) \mid q_\sigma^{(\iota)}(z_k) \leqslant 2M \sum_{k=0}^{\infty} \mid Z_k(x) \mid \mid z_k \mid_t^{(l)} \leqslant M' \mid x \mid_s^{(j)} \quad \text{for some } M', j,$$

the last step following from theorem 1·2. We have also:

Corollary 1. *Let E be an $\mathscr{F}$-space with an absolute base $\{z_n\}$. Then for a basic set to be effective for E in an $\mathscr{F}$-topology $\mathscr{T}^{(\omega)}$ (coarser than $\mathscr{T}_\omega$), it is necessary and sufficient that for each σ there exist M, s such that $q^{(\sigma)}(z_n) \leqslant M \mid z_n \mid_s$ for all n.*

In particular, either from the theorem or from corollary 1,

Corollary 2. *For a basic set $\{p_k\}$ on an $\mathscr{F}$-space E with an absolute base $\{z_n\}$ to be effective for E in its topology $\mathscr{T}_\omega$, it is necessary and sufficient that, given σ, there exist M, s such that $q_\sigma(z_n) \leqslant M \mid z_n \mid_s$ for all n.*

The hypothesis that $\{z_n\}$ is an absolute base cannot simply be omitted from these results; for example, the condition of corollary 2 is insufficient for effectiveness in the Hilbert space of functions regular in $D(1)$ with boundary values in L^2.

8. Absolute effectiveness

A basic set has been defined to be absolutely effective for E in $\mathscr{T}$ if it is effective for E in $\mathscr{T}$ and the basic representations converge absolutely. Instead of the expression $q_\sigma^{(\iota)}(x)$ consider

$$\bar{q}_\sigma^{(\iota)}(x) = \sum_{k=0}^{\infty} \mid \Pi_k(x) \mid \mid p_k \mid_\sigma^{(\iota)}. \tag{8·1}$$

When this is defined (and finite) for all $x \in F_s$, it is a semi-norm on F_s producing a topology $\bar{\tau}_\sigma^{(\iota)}$; if this topology is defined for all ι, we denote the upper-bound topology by $\bar{\tau}_\sigma^{(\omega)}$. We have analogously:

Theorem 8·1. *For a basic set $\{p_k\}$ to be absolutely effective for F in $\mathscr{T}_+$, it is necessary and sufficient that for each s there exists σ such that $\bar{\tau}_\sigma^{(\omega)}$ be defined and coarser than $\mathscr{T}_s^{(\omega)}$; or equivalently that, given s, we can find σ such that for each ι there exist M, j such that $\bar{q}_\sigma^{(\iota)}(x) \leqslant M \mid x \mid_s^{(j)}$ for all $x \in F_s$.*

Proof. The equivalence and necessity follow as in theorem 7·2 with the obvious changes. The sufficiency is immediate.

Theorem 8·2. *If $\{z_n\}$ is an absolute base for F, then for $\{p_k\}$ to be absolutely effective for F in $\mathscr{T}_+$ it is sufficient* (also necessary) *that for each s we can find σ such that for each ι there exist M, j such that $\bar{q}_\sigma^{(\iota)}(z_n) \leqslant M \mid z_n \mid_s^{(j)}$ for all n.*

Proof. Let $\mathscr{T}_\iota^{(\omega)}$ be a topology in which every $x \in F_s$ is represented absolutely by the base $\{z_n\}$. By hypothesis we can find σ such that for each ι there exist M, j such that

$$\bar{q}_\sigma^{(\iota)}(z_n) \leqslant M \,|\, z_n \,|_\iota^{(j)}$$

for all n. Hence

$$\sum_{k=0}^{\infty} |\, \Pi_k(x) \,|\, |\, p_k \,|_\sigma^{(\iota)} \leqslant \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |\, \Pi_k(z_n)\, \mathrm{Z}_n(x) \,|\, |\, p_k \,|_\sigma^{(\iota)} = \sum_{n=0}^{\infty} |\, \mathrm{Z}_n(x) \,|\, \bar{q}_\sigma^{(\iota)}(z_n)$$

$$\leqslant M \sum_{n=0}^{\infty} |\, \mathrm{Z}_n(x) \,|\, |\, z_n \,|_\iota^{(j)} < \infty.$$

We could also state the analogues of the corollaries to theorems 7·2 and 7·3.

The hypothesis that $\{z_n\}$ is an absolute base is obviously essential here, even in a space which has an absolute base; for otherwise it would imply that every base is an absolute base.

Other kinds of convergence could be considered; for instance, unconditional convergence would be straightforward.

9. Uniqueness of basic representations: bases

Basic representations are not, in general, the only possible representations of the form (6·2). It is, however, often important to have unique representations (this justifies, for example, 'equating coefficients'); we therefore now look for a further restriction on a basic set which will secure this. Suppose $\{p_k\}$ is a basic set, effective for the $\mathscr{F}_+$-space F in its topology $\mathscr{T}_+$. Then in particular we have

$$p_k = \sum_{n=0}^{\infty} \Pi_n(p_k)\, p_n \quad (\mathscr{T}_+). \tag{9·1}$$

But on the other hand, we have

$$p_k = \sum_{n=0}^{\infty} \delta_k^n p_n, \tag{9·2}$$

convergent in any topology since it is a finite sum. If the representations are to be unique, we must have

$$\Pi_n(p_k) = \delta_k^n, \tag{U}$$

i.e. the sets $\{\Pi_n\}$ and $\{p_k\}$ must be biorthogonal. A theorem of Iyer (1950, th. 5) indicates that when (U) holds, the $\{p_k\}$ will be suitably independent. We therefore make the following definition:

A basic set $\{p_k\}$ satisfying (U) will be called a U-basic set.

Let us denote the matrix $(\mathrm{Z}_n(p_k))$ by P and the matrix $(\Pi_k(z_n))$ by Π. Then if we apply Z_l to the equation (β), we obtain

$$\sum_{k=0}^{\infty} \mathrm{Z}_l(p_k)\, \Pi_k(z_n) = \delta_n^l, \tag{9·3}$$

or

$$P\Pi = I. \tag{9·4}$$

Referring now to the definition (6·3) of $\Pi_k(p_l)$, we see that (U) reads

$$\sum_{n=0}^{\infty} \Pi_k(z_n)\, \mathrm{Z}_n(p_l) = \delta_k^l, \tag{9·5}$$

or

$$\Pi P = I. \tag{9·6}$$

The condition for a U-basic set is therefore equivalent to the requirement that the matrix Π should be a two-sided inverse of the matrix P. (Note that (9·4) does not imply (9·6), since infinite matrices are in general non-associative.)

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We now justify our definition.

THEOREM 9·1. *Let $\{p_k\}$ be a U-basic set on an $\mathscr{F}_+$-space F and suppose that, for some n, Π_n is defined on F. Then if*

$$\sum_{k=0}^{\infty} \alpha_k p_k = x \quad (\mathscr{T}_+), \qquad \alpha_n = \Pi_n(x).$$

Proof. For some s, $\sum_{k=0}^{\infty} \alpha_k p_k = x \ (\mathscr{T}_s^{(\omega)})$, and $\Pi_n \in F_s^*$. Hence by (U),

$$\Pi_n(x) = \sum_{k=0}^{\infty} \alpha_k \Pi_n(p_k) = \alpha_n.$$

Corollary. If a U-basic set is effective for F in $\mathscr{T}_+$ then the representations are unique.

We now show that, conversely, unique representations are necessarily basic.

THEOREM 9·2. *If the $\mathscr{F}$-space E has a base $\{z_n\}$ and if every $x \in E$ is represented uniquely in an $\mathscr{F}$-topology $\mathscr{T}^{(\omega)}$, coarser than $\mathscr{T}_\omega$, by a series of the form $\sum_{k=0}^{\infty} \alpha_k p_k$, then $\{p_k\}$ is a basic set and the representations are basic.*

Proof. By theorem 1·1, $\alpha_k = \Xi_k(x)$, where $\Xi_k \in E_\omega^*$. Defining $\Pi_k(z_n) = \Xi_k(z_n)$, the set $\{p_k\}$ becomes basic and for any $x \in E$, $\Xi_k(x) = \sum_{n=0}^{\infty} \mathrm{Z}_n(x)\, \Xi_k(z_n) = \Pi_k(x)$ by definition. Similarly:

THEOREM 9·3. *If the $\mathscr{F}_+$-space F has a base $\{z_n\}$ and if every $x \in F$ is represented uniquely in $\mathscr{T}_+$ by a series of the form $\sum_{k=0}^{\infty} \alpha_k p_k$, then $\{p_k\}$ is a U-basic set and the representations are basic.*

The corollary to theorem 9·1 says that if a U-basic set is effective for an $\mathscr{F}_+$-space F in $\mathscr{T}_+$, then it is a base for F. Theorem 9·3 may be interpreted to mean that, given two bases on an $\mathscr{F}_+$-space, the formulae relating the co-ordinates relative to the two bases are those obtainable, as in the algebraic theory, by straightforward formal calculation. This remark emphasizes the fact that in order to change the base, and *a fortiori* in order to discuss basic series, we must have a base to start with. Now contrary to what happens in the algebraic theory, it is false in general that every linear topological space has a base. We could, in fact, have proved theorem 1·1 without the hypothesis that the topology is locally convex, by using $\max_{\mu,\nu} d\left(\sum_{k=\mu}^{\nu} \Xi_k(x)\, x_k, 0\right)$ instead of $\max_{\mu,\nu} \left|\sum_{k=\mu}^{\nu} \Xi_k(x)\, x_k\right|^{(s)}$; while, on the other hand, there exist linear metric spaces on which there are no non-zero continuous linear functionals (Banach 1932, p. 234; Day 1940). The problem of existence of bases in locally convex spaces has not, to the author's knowledge, been solved even for separable Banach spaces over an Archimedean-valued field. It will therefore perhaps be worth while to characterize bases in $\mathscr{F}_+$-spaces.

To this end, a lead is given by the condition for effectiveness of a basic set. Let $\{z_n\}$ be a base for an $\mathscr{F}_+$-space F. Then by theorem 2·2 every F_s is represented uniquely in some $\mathscr{T}_\sigma^{(\omega)}$. By theorem 1·1, $\mathrm{Z}_n \in F_s^*$ and $\mathrm{Z}_n(z_k) = \delta_k^n$; and from the proof of that theorem we have also that $\tau_\sigma^{(\omega)}$ is coarser than $\mathscr{T}_s^{(\omega)}$, where $\tau_\sigma^{(\omega)}$ is the topology defined by the semi-norms $\max_{\mu,\nu} \left|\sum_{k=\mu}^{\nu} \mathrm{Z}_k(x)\, z_k\right|_\sigma^{(\iota)}$; obviously also the closed linear hull $L(\{z_n\})$ of $\{z_n\}$ for $\mathscr{T}_\sigma^{(\omega)}$ contains F_s.

THEOREM 9·4. *Necessary and sufficient conditions for a sequence $\{x_k\}$ of points of an $\mathscr{F}_+$-space F to form a base for F are:*

(i) *given s, we can find t such that $L(\{x_k\})$ for $\mathscr{T}_t^{(\omega)}$ contains F_s,*

(ii) *there is a sequence $\{\Xi_n\} \subset F_s^*$ for all s, orthonormal to $\{x_k\}$,*

(iii) *given t, we can find σ such that for each ι there exist M, j such that*

$$\max_{\mu,\nu}\left|\sum_{k=\mu}^{\nu} \Xi_k(x)\, x_k\right|_\sigma^{(\iota)} \leqslant M\,|\,x\,|_t^{(j)}.$$

In condition (ii), we mean, of course, that Ξ_n is a linear functional on F and that its restriction to F_s is continuous in the topology $\mathscr{T}_s^{(\omega)}$.

Proof. We have only to prove the sufficiency. Given s, choose t as in (i), then σ as in (iii). Writing $S_n(x) = \sum_{k=0}^{n} \Xi_k(x)\, x_k$, we then have in particular, $|\,S_n(x)\,|_\sigma^{(\iota)} \leqslant M\,|\,x\,|_t^{(j)}$ showing that S_n is a continuous linear mapping of the space F_s with topology $\mathscr{T}_t^{(\omega)}$ into the space F_σ with topology $\mathscr{T}_\sigma^{(\omega)}$. Also $\{S_n\}$ converges, by (ii), on the everywhere dense set of linear combinations of the x_k. By Bourbaki (1949, chap. x, § 3, prop. 4), $\{S_n\}$ converges everywhere in F_s. Its limit S is continuous in $\mathscr{T}_t^{(\omega)}$ and coincides on the everywhere dense set with the identical automorphism of F_s. We conclude that $S(x) = x$ for all $x \in F_s$, and s is arbitrary.

For absolute bases we cannot hope to relax the conditions as in the case of basic series (theorem 8·2). The foregoing does, however, emphasize the corresponding inequality (theorem 1·2, corollary 2), to which attention has already been drawn (§ 3).

Finally, we observe that though the definition (6·3) of basic coefficients depends on the base $\{z_n\}$ used, the uniqueness theorem shows that if a U-basic set is effective for F in $\mathscr{T}_+$, the set is U-basic with respect to every base for F and the basic coefficients are independent of the choice of base. This is true even if $\{p_k\}$ is not U-basic, since if $\{z'_m\}$ is any other base for F, $z'_m = \sum_{k=0}^{\infty} \Pi_k(z'_m)\, p_k$ $(\mathscr{T}_+)$ by the effectiveness, and by theorem 6·1,

$$\Pi_k(x) = \Pi_k\left(\sum_{m=0}^{\infty} Z'_m(x)\, z'_m\right) = \sum_{m=0}^{\infty} Z'_m(x)\, \Pi_k(z'_m) = \Pi'_k(x).$$

10. Application to functions regular in circles

Consider the $\mathscr{F}_+$-space $H(\rho)$ $(0<\rho\leqslant\infty)$ or $\bar{H}(R)$ $(0\leqslant R<\infty)$ with its usual topology (§§ 3, 4), and the Taylor base $\{z^n\}$. In accordance with § 6, a basic set on $H(\rho)$ (or $\bar{H}(R)$) is a set $\{p_k(z)\}$, regular and not identically zero in $D(\rho)$ (or $\overline{D(R)}$), for which we are given a matrix (π_{nk}) of coefficients with the property

$$z^n = \sum_{k=0}^{\infty} \pi_{nk}\, p_k(z) \qquad (10·1)$$

uniformly convergent in every $\overline{D_-(\rho)}$ (or in some $\overline{D_+(R)}$).

The basic series of a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is now the series $\sum_{k=0}^{\infty} \Pi_k f(0)\, p_k(z)$, where $\Pi_k f(0) = \sum_{n=0}^{\infty} a_n \pi_{nk}$, and the problem that of representing each $f \in H(\rho)$ (or $\bar{H}(R)$) in the corresponding topology by its basic series. We shall abbreviate 'represents f in the topology of $H(\rho)$' to 'represents f in $D(\rho)$' and 'represents in the topology of $\bar{H}(R)$' to 'represents in

$D_+(R)$'. We shall also use 'represents in $\overline{D(R)}$' to mean 'represents in the topology induced by the norm $\max_{|z|=R} |f(z)|$ on $\bar{H}(R)$'; the convergence here is, of course, uniform convergence in $\overline{D(R)}$. When the representations are basic, we shall say, for example, effective for $H(\rho)$ in $\overline{D(R)}$.

The semi-norms (they are actually norms) in the definition of the topology of $H(\rho)$ may be taken as $\max_{|z|=r} |f(z)|$, where r is one of a sequence $\{r_s\}$ increasing to ρ; then $q_s(z_n)$ may be written

$$F_n(r) = \max_{\mu,\nu} \max_{|z|=r} \left| \sum_{k=\mu}^{\nu} \pi_{nk} p_k(z) \right|. \tag{10·2}$$

For $\{p_k(z)\}$ to be effective for $H(\rho)$ in $\overline{D(r)}$ it is necessary and sufficient (by theorem 7·3, corollary 1) that there exist M and $R<\rho$ such that for all n, $F_n(r) \leqslant MR^n$. If we write

$$\kappa(r) = \overline{\lim_{n\to\infty}} [F_n(r)]^{1/n}, \tag{10·3}$$

we obtain from this $\kappa(r) \leqslant R < \rho$. Conversely, if $\kappa(r)<\rho$, there is an $R<\rho$ such that $F_n(r) \leqslant R^n$ for large n. Since $F_n(r) < \infty$ by (10·1),

THEOREM 10·1. *For a basic set $\{p_k(z)\}$ (on $\bar{H}(r)$) to be effective for $H(\rho)$ in $\overline{D(r)}$ it is necessary and sufficient that $\kappa(r)<\rho$.*

Similarly, the conditions for effectiveness in other cases may be expressed more concisely in terms of the Cannon function $\kappa(r)$. If we observe that

$$F_n(r) \geqslant \max_{|z|=r} \left| \sum_{k=0}^{\infty} \pi_{nk} p_k(z) \right| = r^n,$$

we have
$$\kappa(r) \geqslant r. \tag{10·4}$$
From this we obtain easily:

THEOREM 10·2. *For a basic set $\{p_k(z)\}$ to be effective for (a) $H(\rho)$ in $D(\rho)$, (b) $\bar{H}(R)$ in $D_+(R)$ or (c) $\bar{H}(R)$ in $\overline{D(R)}$, it is necessary and sufficient that (a) $\kappa(r)<\rho$ for all $r<\rho$, (b) $\kappa(R+) = R$, (c) $\kappa(R) = R$, where $\kappa(R+) = \lim_{\rho\to R+} \kappa(\rho)$.*

The limit exists since $\kappa(R)$ is monotone increasing. As usual, we can have $\rho = \infty$ in (a) or $R = 0$ in (b).

In the case of absolute effectiveness we define, similarly:

$$M_k(r) = \max_{|z|=r} |p_k(z)|, \tag{10·5}$$

$$\omega_n(r) = \sum_{k=0}^{\infty} |\pi_{nk}| M_k(r), \tag{10·6}$$

$$\lambda(r) = \overline{\lim_{n\to\infty}} [\omega_n(r)]^{1/n}. \tag{10·7}$$

From theorem 8·2 we now readily obtain:

THEOREM 10·3. *For a basic set $\{p_k(z)\}$ to be absolutely effective for (a) $H(\rho)$ in $D(r)$, (b) $H(\rho)$ in $D(\rho)$, (c) $\bar{H}(R)$ in $D_+(R)$ or (d) $\bar{H}(R)$ in $\overline{D(R)}$, it is necessary and sufficient that (a) $\lambda(r)<\rho$, (b) $\lambda(r)<\rho$ for all $r<\rho$, (c) $\lambda(R+) = R$ or (d) $\lambda(R) = R$.*

Whittaker (1949) defines a basic set of polynomials to be a basic set $\{p_k(z)\}$ such that the representation (10·1) is a unique finite sum; we shall call such a basic set a *W-basic set*. With

$$p_k(z) = \sum_l p_{kl} z^l \tag{10·8}$$

the matrix (p_{kl}) is row-finite; Whittaker (1949, p. 40, T_{31}) has proved:

THEOREM 10·4. *For a set $\{p_k(z)\}$ to be W-basic it is necessary and sufficient that the matrix (p_{kl}) should have a unique row-finite inverse (π_{nk}).*

Since the matrices (p_{kl}) and (π_{nk}) are just the transposes $\tilde{P}$, $\tilde{\Pi}$ of the matrices P, Π of § 9, we have:

Corollary. Every W-basic set is U-basic.

In order to extend the results on W-basic sets to general functions $\{p_k(z)\}$, Nassif (see Whittaker 1949, p. 77) defines what we shall call an *N-basic set* to be a basic set such that the representations (10·1) are unique. This definition is felt to be unsatisfactory for two reasons: the hypothesis of uniqueness is too restrictive, since a W-basic set need not be N-basic, and, moreover, it has led to no new results. Nassif extends to N-basic sets the conditions for effectiveness, which are true without the extra hypothesis, but not the counterpart of theorem 9·1; the question whether this theorem holds for an N-basic set still remains open. For an example of a W-basic set which is not N-basic see Whittaker (1935, p. 16).

From theorem 9·1, corollary, we obtain:

THEOREM 10·5. *If a U-basic set is effective for $H(\rho)$ in $D(\rho)$ or for $\bar{H}(R)$ in $D_+(R)$ then the representations are unique.*

This (with theorem 10·4, corollary) includes the known results on W-basic sets (Whittaker 1949, pp. 62–63, T_{41} and T_{42}). The corresponding proposition for $\bar{H}(R)$ in $\overline{D(R)}$ is false, even for a W-basic set which is also effective for $\bar{H}(R)$ in $D_+(R)$; this is shown by the following example:

$$p_0(z) = 1, \quad p_1(z) = z-1, \quad p_n(z) = n^{-2}z^n-(n-1)^{-2}z^{n-1} \quad \text{for } n>1.$$

For $n>0$, we have $z^n = n^2 \sum_{k=0}^{n} p_k(z)$, and a short calculation gives $\kappa(R) = R$ for $R \geqslant 1$. The set $\{p_k(z)\}$ is therefore effective for $\bar{H}(1)$ in $D_+(1)$ and in $\overline{D(1)}$, whilst $\sum_{k=0}^{\infty} p_k(z)$ represents zero in $\overline{D(1)}$.

It can, however, be shown (on the lines of Whittaker 1949, T_{40}) that the convergence of $\sum_{n=0}^{\infty} |\pi_{nk}|^2 R^{-2n}$ for all k is sufficient for effectiveness in $\overline{D(R)}$ to imply uniqueness.

Theorems 2·2 and 9·2 enable us to state:

THEOREM 10·6. *If a basic set represents $H(\rho)$ or $\bar{H}(R)$ uniquely in $D(r)$, $D_+(r)$ or $\overline{D(r)}$, then the representations are basic.*

The uniqueness is essential here; the following example is due to Professor Whittaker:

$$p_0(z) = 1, \quad p_n(z) = z^n-z^{n-1} \text{ for } n>0; \quad z^n = \sum_{k=0}^{n} p_k(z).$$

The set $\{p_k(z)\}$ is therefore W-basic, but is not effective for $H(R)$ in $D(R)$ for any $R \leqslant 1$; in particular, none of the basic coefficients of $(1-z)^{-1}$ converges. However, it is easily verified that $\sum_{n=0}^{\infty}\left(\lambda-\sum_{k=0}^{n-1} a_k\right) p_k(z)$ represents any $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(R)$ in $D(R)$ for all λ.

This same example illustrates how a set can be basic with different Π-matrices (cf. § 6); here we can take, for any λ,

$$\pi_{nk} = \begin{cases} \lambda & (k \leqslant n) \\ \lambda - 1 & (k > n). \end{cases}$$

Theorems 10·5 and 10·6 enable us to complete the proof given by Boas (1948, lemma 2), that if $\{p_n(z)\}$, $\{p_n^*(z)\}$ are two simple sets (i.e. if $p_n(z)$, $p_n^*(z)$ are polynomials of degree n) such that $p_n(z) = p_n^*(z)$ for $n > N$, and if $\{p_n^*(z)\}$ is effective for $H(R)$ in $D(R)$ or for $\bar{H}(R)$ in $\overline{D(R)}$, then $\{p_n(z)\}$ has the same property. For by hypothesis each $f \in H(R)$ (or $\bar{H}(R)$) is represented in $D(R)$ (or in $D_+(R)$, by Whittaker 1949, T_{12}) by its basic series for the set $\{p_n^*(z)\}$, and the representations are unique by theorem 10·5. Hence, as in Boas's proof, each such f is so represented by a unique series $\sum_{n=0}^{\infty} c_n p_n(z)$ and theorem 10·6 applies. The hypothesis that the sets are simple could be weakened.

It may be thought that a U-basic set consisting of polynomials would necessarily be W-basic, but this is not the case. The set $\{p_n(z)\}$, where $p_n(z) = (z^{n+1}/(n+1)!) - (z^n/n!)$, is U-basic with $z^n = -\sum_{k=n}^{\infty} n!\,p_k(z)$ in $D(\infty)$, but it is not W-basic; moreover, it is easy to show that $\kappa(R) = R$ for all R, so that it is everywhere effective.

We have seen that representations can be unique in $D_+(R)$ without being unique in $\overline{D(R)}$; in other words, that a basic set can represent zero in $\overline{D(R)}$ but in no $D_+(R)$. A basic set can also represent zero in every $\overline{D(r)}$ $(r < R)$ without necessarily giving a representation in $D(R)$. For example, $2\alpha_{nk} = (n+k+1)^2 - (n-k+1)$ defines a $(1, 1)$ correspondence between non-negative integers and ordered pairs (n, k) of non-negative integers. Define

$$\left.\begin{aligned} p_{\alpha_{nk}}(z) &= \left(\frac{n+2}{n+1}z\right)^{\alpha_{nk}} - \left(\frac{n+2}{n+1}z\right)^{\alpha_{n,k-1}} \quad \text{if} \quad k > 0, \\ p_{\alpha_{n0}}(z) &= \left(\frac{n+2}{n+1}z\right)^{\alpha_{n0}}. \end{aligned}\right\} \tag{i}$$

Then $\sum_{k=0}^{m} p_{\alpha_{nk}}(z) = \left(\frac{n+2}{n+1}z\right)^{\alpha_{nm}}$, and for any fixed n this tends uniformly to zero as $m \to \infty$ in any closed region interior to $|z| < \frac{n+1}{n+2}$, i.e. if $r < 1$, there is an expansion of zero in $\overline{D(r)}$. Suppose now that

$$\sum_{k=0}^{\infty} \beta_k p_k(z) \tag{ii}$$

represents zero in $D(1)$. From (i) we find by differentiation

$$p_{\alpha_{nk}}^{(m)}(0) = \begin{cases} \left(\dfrac{n+2}{n+1}\right)^m m! & \text{if} \quad m = \alpha_{nk}, \\ -\left(\dfrac{n+2}{n+1}\right)^m m! & \text{if} \quad m = \alpha_{n,k-1}, \\ 0 & \text{otherwise}. \end{cases}$$

Hence if we differentiate (ii) $m = \alpha_{nk}$ times, we obtain, on putting $z = 0$,

$$\beta_{\alpha_{nk}}\left(\frac{n+2}{n+1}\right)^m m! - \beta_{\alpha_{n,k+1}}\left(\frac{n+2}{n+1}\right)^m m! = 0,$$

i.e. $\beta_{\alpha_{n,k+1}} = \beta_{\alpha_{nk}}$. Hence if, for some $m = \alpha_{nk}$, $\beta_m \neq 0$, the series (ii) cannot converge uniformly to zero in $D\left(\frac{n+1}{n+2}\right)$, since on that circle $\max|p_{\alpha_{nl}}(z)| = 2$ for all l. It follows that $\beta_m = 0$ for all m and there is no expansion of zero in $D(1)$.

A W-basic set is called *C-basic* if it satisfies Cannon's condition

$$\lim_{n\to\infty} N_n^{1/n} = 1, \tag{C}$$

where N_n is the number of non-zero π_{nk} in (10·1). Since

$$F_n(R) \leqslant \omega_n(R) \leqslant N_n \max_k |\pi_{nk}| M_k(R) \leqslant N_n F_n(R),$$

we have $\lambda(R) = \kappa(R)$ for a C-basic set and effectiveness implies absolute effectiveness. We can, however, have $\lambda(R) = \kappa(R)$ for a set which is not C-basic (see e.g. Cannon 1937).

11. The Cannon functions

Having established the connexion between effectiveness or absolute effectiveness and the corresponding Cannon function $\kappa(r)$ or $\lambda(r)$, it is natural to investigate the properties of these functions.

Theorem 11·1. $\kappa(R^{1+a}) \leqslant [\kappa(R)]^{1-(a/b)} [\kappa(R^{1+b})]^{a/b}$ $(0 < a/b < 1)$.

For a proof for W-basic sets see Whittaker (1949, p. 37, T_{26}). See also theorem 11·5 below.

Theorem 11·2. *There is at most one interval in which $\kappa(R)$ is constant, the left-hand end-point of which is $R = 0$. $\kappa(R)$ increases strictly from a certain value onwards.*

This follows from theorem 11·1 as in Whittaker (1949, p. 38, T_{28}, and p. 17, T_9).

Theorem 11·3. *$\kappa(R)$ has at most one discontinuity in $0 < R < \infty$; in fact, $\kappa(R-) < \kappa(R+)$ implies $\kappa(R+) = \infty$.*

Proof.† This follows easily from Hardy, Littlewood & Pólya (1934, §111, p. 91), since $\log\kappa(r)$ is a convex function of $\log r$.

Using (10·4), we find also (ibid. §91, p. 75):

Lemma. If $\kappa(R-) = R$ for two values R_1, R_2 with $0 < R_1 < R_2 < \infty$, then $\kappa(r) = r$ for $R_1 \leqslant r < R_2$.

Theorem 11·4. *The values of R such that a basic set is effective for $H(R)$ in $D(R)$ make up an interval $a \leqslant R \leqslant b$ or $a < R \leqslant b$, together, possibly, with the single point $R = \infty$.*

The values of R such that a basic set is effective for $\bar{H}(R)$ in $D_+(R)$ make up a single interval $a \leqslant R \leqslant b$ or $a \leqslant R < b$, together, possibly, with the single point $R = 0$.

The values of R such that a basic set is effective for $\bar{H}(R)$ in $\overline{D(R)}$ make up an interval $a \leqslant R \leqslant b$ or $a \leqslant R < b$.

In all cases the end-points a, b may be 0, ∞ or equal to each other. Most of the theorem follows from the lemma, since $\kappa(R-) = R$ is necessary for effectiveness in all three cases. The rest is obvious.

Examples showing that the various possibilities left open actually can occur will be found in Whittaker (1949, pp. 12–14). To complete the list, we give the following two examples, which were also supplied by Professor Whittaker:

Example 1. $p_n(z) = z^{2k+1} + z^{4k} + 2^{k^2} z^{2k^3+1}$ if $n = 4k$,

$p_n(z) = z^n$ otherwise.

† I owe this remark to the referee; my original proof was direct.

It will be found that
$$\kappa(R) = \begin{cases} \sqrt{R} & \text{if} \quad R<1, \\ \infty & \text{if} \quad R \geqslant 1, \end{cases}$$

so that the W-basic set is effective in the single open circle $D(1)$ (and at the origin).

Example 2.
$$p_n(z) = z^{2k+1}+z^{4k}+z^{2k^2+1} \quad \text{if} \quad n=4k,$$
$$p_n(z) = z^n \quad \text{otherwise.}$$

It will now be found that
$$\kappa(R) = \begin{cases} \sqrt{R} & \text{if} \quad R \leqslant 1, \\ \infty & \text{if} \quad R>1, \end{cases}$$

so that the W-basic set is here effective in $D(1)$ and in $\overline{D(1)}$ (and at the origin), but in no other circle.

The function $\lambda(r)$ has been considered hitherto only for C-basic sets, but the corresponding results hold generally.

THEOREM 11·5. $\lambda(R^{1+a}) \leqslant [\lambda(R)]^{1-(a/b)}\,[\lambda(R^{1+b})]^{a/b} \quad (0<a/b<1).$

Proof. With $M(R) = \max\limits_{|z|=R} |f(z)|$, we use Hadamard's three circles theorem in the form
$$M(R^{1+a}) \leqslant [M(R)]^{1-(a/b)}\,[M(R^{1+b})]^{a/b} \quad (0<a/b<1), \tag{i}$$

and Hölder's inequality in the form
$$\sum_{k=0}^{\infty} a_k^{1/r}\, b_k^{1/r'} \leqslant \left[\sum_{k=0}^{\infty} a_k\right]^{1/r} \left[\sum_{k=0}^{\infty} b_k\right]^{1/r'}, \tag{ii}$$

where a_k, $b_k \geqslant 0$, $r>1$ and $(1/r)+(1/r')=1$. Applying (i) to the function $f(z) = \pi_{nk}\, p_k(z)$,
$$|\pi_{nk}|\, M_k(R^{1+a}) \leqslant [|\pi_{nk}|\, M_k(R)]^{1-(a/b)}\,[|\pi_{nk}|\, M_k(R^{1+b})]^{a/b}.$$

Summing now from $k=0$ to ∞ and applying (ii) to the right-hand side with
$$a_k = |\pi_{nk}|\, M_k(R), \quad b_k = |\pi_{nk}|\, M_k(R^{1+b}), \quad \text{and} \quad 1/r = 1-(a/b),$$
$$\omega_n(R^{1+a}) \leqslant [\omega_n(R)]^{1-(a/b)}\,[\omega_n(R^{1+b})]^{a/b},$$

and the result follows by taking nth roots and upper limits as $n \to \infty$.

Theorems 11·2, 11·3 and 11·4 also hold with $\lambda(R)$ for $\kappa(R)$ and 'absolutely effective' for 'effective'.

12. Alternative conditions for effectiveness in circles

We have seen in theorem 6·1 that if a basic set $\{p_k(z)\}$ is effective anywhere, then Π_k is a continuous linear functional on the space concerned. Using theorem 3·1 we may therefore write
$$\Pi_k f(0) = \frac{1}{2\pi \mathrm{i}} \int_C f(z)\, \pi_k(z)\, \mathrm{d}z, \tag{12·1}$$

where
$$\pi_k(z) = \sum_{n=0}^{\infty} \pi_{nk}\, z^{-n-1}, \tag{12·2}$$

and C is a circle such that $f(z)$ is regular in $\overline{D(C)}$ and $\pi_k(z)$ in $\overline{E(C)}$. (The existence of this circle is a consequence of the continuity and hence of the existence of $\Pi_k f(0)$ for all f in some $H(\rho)$.) We may state:

THEOREM 12·1. *In order that* $\Pi_k f(0)$ *shall exist for all* $f \in H(\rho)$ *or* $\bar{H}(\rho)$, *it is necessary and sufficient that* $\overline{\lim_{n\to\infty}} |\pi_{nk}|^{1/n} < \rho$ *or* $\overline{\lim_{n\to\infty}} |\pi_{nk}|^{1/n} \leqslant \rho$ *respectively* (cf. Boas 1949, lemma 1, and Iyer 1948, lemma).

This theorem enables us to give an upper bound for the radius of circles in which there can be an expansion of zero, namely, $\kappa(0+)$; for it shows (with Whittaker 1949, p. 62, equation (XI.8)) that $\Pi_k f(0)$ is defined on $\bar{H}(\kappa(0+))$ and theorem 9·1 applies.

In particular, (12·1) gives

$$\pi_{nk} = \frac{1}{2\pi i}\int_C z^n \pi_k(z)\,dz, \tag{12·3}$$

and for a U-basic set $\{p_k(z)\}$

$$\frac{1}{2\pi i}\int_C p_n(z)\,\pi_k(z)\,dz = \delta_k^n, \tag{12·4}$$

which is equivalent to the condition (U) for a U-basic set. Obviously

$$|\Pi_k f(0)| \leqslant R \max_{|z|=R} |f(z)| \max_{|z|=R} |\pi_k(z)|, \tag{12·5}$$

which is the analogue of Cauchy's inequality.

THEOREM 12·2. *For any* $\rho > \kappa(r)$,

$$(w-z)^{-1} = \sum_{k=0}^{\infty} p_k(z)\,\pi_k(w), \tag{12·6}$$

where the series converges uniformly for $z \in \overline{D(r)}$, $w \in \overline{E(\rho)}$.

This may be proved by a standard argument (cf. Titchmarsh 1939, pp. 30–31) for the reversal of the order of summation in the repeated series $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} w^{-n-1}\pi_{nk}\,p_k(z)$. Integrating term by term and using (12·1) we recover the sufficiency of the condition for effectiveness.

THEOREM 12·3. *For* $\{p_k(z)\}$ *to be effective for* $H(\rho)$ *in* $\overline{D(r)}$ *it is necessary and sufficient that* (12·6) *holds uniformly for* $z \in \overline{D(r)}$, $w \in \overline{E(R)}$ *for some* $R < \rho$.

The necessity is immediate from theorems 10·1 and 12·2. The sufficiency follows by integration as indicated above. Similarly:

THEOREM 12·4. *For* $\{p_k(z)\}$ *to be effective for (a)* $H(R)$ *in* $D(R)$, *(b)* $\bar{H}(R)$ *in* $\overline{D(R)}$, *or (c)* $\bar{H}(R)$ *in* $D_+(R)$, *it is necessary and sufficient that* (12·6) *holds uniformly for (a)* $z \in \overline{D(r)}$, $w \in \overline{E(r_1)}$ *for all* $r < R$ *and some* $r_1(r) < R$, *(b)* $z \in \overline{D(R)}$, $w \in \overline{E(\rho)}$ *for all* $\rho > R$, *or (c)* $z \in \overline{D(r)}$, $w \in \overline{E(\rho)}$ *for all* $\rho > R$ *and some* $r(\rho) > R$.

It is interesting to note that for $\{p_k(z)\}$ to be effective for $H(\rho)$ in $\overline{D(r)}$, $(w-z)^{-1}$ must be represented by its basic series for some w with $|w| < \rho$, i.e. functions not in $H(\rho)$ must be represented.

THEOREM 12·5. *For* $\{p_k(z)\}$ *to be absolutely effective for* $H(\rho)$ *in* $\overline{D(r)}$ *it is necessary and sufficient that* (12·6) *holds for* $z \in \overline{D(r)}$, $w \in \overline{E(R)}$ *for some* $R < \rho$ *and that* $\sum_{k=0}^{\infty} M_k(r) \max_{|z|=R} |\pi_k(z)| < \infty$.

The necessity of the extra condition follows by majorizing the series and using theorem 10·3. The sufficiency is immediate from (12·5). We could also state an analogue for theorem 12·4.

13. Effectiveness in general domains

The general results of §§ 6 to 9 may also be applied to the space $H(D)$ of functions regular in a simply-connected domain D (§ 3). Here, too, the conditions for effectiveness may be expressed more concisely in terms of a Cannon function. In our usual notation, for a basic set $\{p_k(z)\}$ we have

$$[\psi(z)]^n = \sum_{k=0}^{\infty} \pi_{nk}\, p_k(z) \quad (D), \tag{13·1}$$

and for any compact subset A of D we write

$$F_n(A) = \max_{\mu,\nu} \max_{z\in A} \left| \sum_{k=\mu}^{\nu} \pi_{nk} p_k(z) \right|, \tag{13·2}$$

$$\kappa(A) = \overline{\lim_{n\to\infty}}\, [F_n(A)]^{1/n}. \tag{13·3}$$

For $\{p_k(z)\}$ to be effective for $H(D)$ in D, corollary 2 of theorem 7·3 gives the condition that given A we can find $A' \subset D$ such that $F_n(A) \leqslant M \max_{z\in A'} |\psi(z)|^n$ for some M and all n. This is equivalent to $\kappa(A) < 1$ *for all compact* $A \subset D$.

On account of the isomorphism between the $\mathscr{F}$-spaces $H(D)$ and $H(1)$, their duals are also isomorphic, and can therefore be identified with $\bar{H}(1)$. It would be interesting to know if they are also isomorphic with $\bar{H}(D)$, but the answer is only obvious when D is bounded by the regular curve C. In the latter case, the function ψ conformally maps C on to the unit circle and a neighbourhood of $\overline{D(C)}$ on to a neighbourhood of $\overline{D(1)}$, thus giving the required isomorphism. It follows easily that for the basic set $\{p_k(z)\}$ to be effective for $\bar{H}(C)$ in $D_+(C)$ or in $\overline{D(C)}$ it is necessary and sufficient that

$$\kappa(C+) = 1 \quad \text{or} \quad \kappa(C) = 1.$$

The case of absolute effectiveness is exactly similar. The uniqueness theorems apply directly.

For the case of $H(C)$ we can obtain more detailed results by using the Faber polynomials of C. Let χ map $D(1)$ in the t-plane conformally on to $E(C)$ in the z-plane so that

$$z = \chi(t), \quad \chi(0) = \infty. \tag{13·4}$$

Then χ will be regular, save for the pole at the origin, in some neighbourhood of $\overline{D(1)}$. We shall denote by γ_r a circle of radius r which lies inside this neighbourhood; its image C_r lies inside or outside C, according as $r > 1$ or $r < 1$. It was shown by Faber (1903) that

$$\frac{t\chi'(t)}{\chi(t) - z} = \sum_{n=0}^{\infty} P_n(z)\, t^n, \tag{13·5}$$

where $P_n(z)$ is a polynomial of degree n; this is the nth Faber polynomial for C. Moreover,

$$\lim_{n\to\infty} \max_{z\in \overline{D(C_r)}} |P_n(z)|^{1/n} = 1/r. \tag{13·6}$$

Faber also showed that every $f \in H(C_r)$ is represented uniquely in $D(C_r)$ by the series

$$f(z) = \sum_{n=0}^{\infty} a_n P_n(z), \tag{13·7}$$

where

$$a_n = \frac{1}{2\pi i} \int_{\gamma_R} f(\chi(t))\, t^{n-1}\, dt \quad (R > r), \tag{13·8}$$

and

$$\overline{\lim_{n\to\infty}}\, |a_n|^{1/n} \leqslant r. \tag{13·9}$$

This shows that the Faber polynomials form an absolute base for $H(C_r)$ and for $\bar{H}(C_r)$. The Faber polynomials for the curve C_r are easily seen to be the multiples $r^n P_n(z)$ of those for C.

From (13·7) and (13·8), we see in particular that

$$\sum_{n=0}^{\infty} P_n(z)\, P_{-n-1}(w) = (w-z)^{-1}, \tag{13·10}$$

where
$$P_{-n-1}(w) = \frac{1}{2\pi \mathrm{i}}\int_{\gamma_R} \frac{t^{n-1}\,\mathrm{d}t}{w-\chi(t)} \tag{13·11}$$

and $w \in \overline{E(C_r)}$. Since $\lim_{n\to\infty} \max_{w\in \overline{E(C_r)}} |P_{-n-1}(w)|^{1/n} \leqslant r$, we see that the series (13·10) converges uniformly for $z \in \overline{D(C_R)}$ and $w \in \overline{E(C_r)}$. The notation has been chosen by analogy with the functions z^{-n-1} associated with the Taylor base $\{z^n\}$. We have, further,

$$\frac{1}{2\pi \mathrm{i}}\int_{C_r} P_k(z)\, P_{-n-1}(z)\,\mathrm{d}z = \delta_n^k. \tag{13·12}$$

For the left-hand side is

$$(2\pi\mathrm{i})^{-1}\int_{\gamma_R} t^{n-1}\,\mathrm{d}t\,(2\pi\mathrm{i})^{-1}\int_{C_r} P_k(z)\,(z-\chi(t))^{-1}\,\mathrm{d}z$$
$$= (2\pi\mathrm{i})^{-1}\int_{\gamma_R} t^{n-1} P_k(\chi(t))\,\mathrm{d}t \quad \text{by Cauchy's integral,}$$
$$= \delta_k^n \quad \text{by (13·8).}$$

From this it follows that:

Theorem 13·1. *Every* $\Phi \in H^*(C)$ *is of the form* $\Phi(f) = (2\pi\mathrm{i})^{-1}\int_\Gamma f(z)\,\phi(z)\,\mathrm{d}z$, *where* $\phi(z)$ *is regular in* $\overline{E(C)}$, $\phi(\infty) = 0$, *and* Γ *is any curve such that* f *is regular in* $\overline{D(\Gamma)}$ *and* ϕ *in* $\overline{E(\Gamma)}$.

For by theorem 1·3, $\Phi(f) = \sum_{n=0}^{\infty} a_n c_n$, where $f(z) = \sum_{n=0}^{\infty} a_n P_n(z)$ and $\overline{\lim_{n\to\infty}}\, |c_n|^{1/n} < 1$. It is sufficient to write $\phi(z) = \sum_{n=0}^{\infty} c_n P_{-n-1}(z)$ and integrate term by term.

After these preliminaries it is now easy not only to apply the results of §§ 7, 8 to $H(C)$ and $(\bar{H}C)$ with Faber polynomials as base, but also to extend the results of § 12. We now have

$$P_n(z) = \sum_{k=0}^{\infty} \tilde{\omega}_{nk}\, p_k(z), \tag{13·13}$$

and from (13·6) the condition for effectiveness for $H(C)$ in $D(C)$ becomes $\kappa(C_r) < 1$ for all $r > 1$ (where κ now refers to the representation (13·13)). The conditions for effectiveness for $(\bar{H}C)$ in $D_+(C)$ and in $\overline{D(C)}$ are similarly $\kappa(C+) = 1$ and $\kappa(C) = 1$. The latter has also been obtained by Elnabi (see Whittaker 1949, pp. 77, 78).

From theorem 13·1 we see that $\Pi_k f(C)$ (in Elnabi's notation) is given by

$$\Pi_k f(C) = (2\pi\mathrm{i})^{-1}\int_{C_r} f(z)\,\tilde{\omega}_k(z)\,\mathrm{d}z, \tag{13·14}$$

where
$$\tilde{\omega}_k(z) = \sum_{n=0}^{\infty} \tilde{\omega}_{nk} P_{-n-1}(z), \tag{13·15}$$

and the methods of § 12 will give conditions for the existence of $\Pi_k f(C)$ and for effectiveness in terms of the convergence of

$$\sum_{k=0}^{\infty} p_k(z)\,\tilde{\omega}_k(w) = (w-z)^{-1} \tag{13·16}$$

in suitable regions.

Using the fact that $\max_{z\in C_r} |f(z)| = \max_{t\in\gamma_r} |f(\chi(t))|$, we could also extend to the family (C_r) of parallel curves the properties of κ given in § 11.

If we write

$$\Pi_{nk} = (2\pi i)^{-1}\int_{\gamma_r} t^{k-1}[\chi(t)]^n\,dt, \tag{13·17}$$

we have

$$z^n = \sum_{k=0}^{n} \Pi_{nk}\,P_k(z) \tag{13·18}$$

and for large w,

$$P_{-k-1}(w) = (2\pi i)^{-1}\int_{\gamma_r} t^{k-1}(w-\chi(t))^{-1}\,dt$$

$$= (2\pi i)^{-1}\int_{\gamma_r} t^{k-1}\sum_{n=0}^{\infty} w^{-n-1}[\chi(t)]^n\,dt = \sum_{n=0}^{\infty} w^{-n-1}(2\pi i)^{-1}\int_{\gamma_r} t^{k-1}[\chi(t)]^n\,dt,$$

so that, for large w,

$$P_{-k-1}(w) = \sum_{n=0}^{\infty} \Pi_{nk}\,w^{-n-1}. \tag{13·19}$$

This result relates the function $P_{-k-1}(w)$ to the function $\pi_k(w)$ of § 11. Returning to the case of $H(D)$, we shall still have $z^n = \sum_{k=0}^{\infty} \psi_{nk}[\psi(z)]^k$ and may define, formally,

$$\psi_{-k-1}(z) = \sum_{n=0}^{\infty} \psi_{nk}\,z^{-n-1}.$$

It would be interesting to know in what neighbourhood of infinity (if any) this function is regular, and whether any of the properties found for the special cases extend to this more general case.

14. Representation of integral functions

The general results apply also to the space $I(\rho, \sigma)$ of integral functions of increase not exceeding order ρ, type σ $(0<\rho<\infty,\ 0\leqslant\sigma<\infty)$. For $\{p_k(z)\}$ to be effective for $I(\rho,\sigma)$ in $\overline{D(R)}$ it is necessary and sufficient (by theorem 7·3, corollary 1) that there exist M, $r<\sigma^{-1/\rho}$ such that, for all n, $F_n(R)\leqslant M(n/e\rho)^{n/\rho}\,r^n$; we obtain:

Theorem 14·1. *For a basic set $\{p_k(z)\}$ to be effective for $I(\rho,\sigma)$ in $\overline{D(R)}$ it is necessary and sufficient that* $\overline{\lim_{n\to\infty}}\,[(e\rho/n)^{1/\rho}\,(F_n(R))^{1/n}]<\sigma^{-1/\rho}$.

For effectiveness in $D(\infty)$ the condition must hold for all finite R.

Similarly, for the space $\bar{I}(\rho,\sigma)$ of integral functions of increase less than order ρ, type σ $(0<\rho<\infty,\ 0<\sigma\leqslant\infty)$:

Theorem 14·2. *For a basic set $\{p_k(z)\}$ to be effective for $\bar{I}(\rho,\sigma)$ in $\overline{D(R)}$ it is necessary and sufficient that* $\overline{\lim_{n\to\infty}}\,[(e\rho/n)^{1/\rho}\,(F_n(R))^{1/n}]\leqslant\sigma^{-1/\rho}$.

For functions of increase less than order ρ_0 type 0 we must have

$$\overline{\lim_{n\to\infty}}\,[(e\rho/n)^{1/\rho}\,(F_n(R))^{1/n}] = 0$$

for all $\rho<\rho_0$. Similar results could be stated for $D(R)$ and $D_+(R)$. The $\{p_k(z)\}$ need be regular only in the region in which we wish to represent by series. Our results may equivalently be expressed in terms of order and type of the basic set (Whittaker 1949, chap. XII; Eweida 1950).

The uniqueness theorem (9·1, corollary) does not apply since the topology of $H(R)$, even for infinite R, is strictly coarser than that of $I(\rho, \sigma)$.

PART III. BASIC SETS ASSOCIATED WITH GIVEN SETS

15. Definitions and notations

In this part of the paper we shall be concerned almost entirely with representation of regular functions in circles. Since we discuss mainly effectiveness for $H(\rho)$ in $D(\rho)$ or for $\bar{H}(R)$ in $D_+(R)$ or in $\overline{D(R)}$, we shall now use the usual abbreviations *effective in* $D(\rho)$, $D_+(R)$ *or* $\overline{D(R)}$ respectively.

We saw in § 9 that a basic set $\{p_k(z)\}$ determines a matrix (p_{nk}) and a left inverse (π_{nk}); conversely, such a pair of matrices formally determines a basic set. Now with any pair of matrices there are associated in a natural way other pairs of matrices; hence, with any basic set we can associate other sets, and inquire into the relation between the original set and those associated with it.

The *inverse* of a U-basic set is defined to be the set $\{\check{p}_n(z)\}$, where

$$\check{p}_n(z) = \sum_{k=0}^{\infty} \pi_{nk} z^k, \tag{15·1}$$

$$z^n = \sum_{k=0}^{\infty} p_{nk} \check{p}_k(z). \tag{15·2}$$

The *transpose* of a U-basic set is defined to be the set $\{\tilde{p}_n(z)\}$, where

$$\tilde{p}_n(z) = \sum_{k=0}^{\infty} p_{kn} z^k, \tag{15·3}$$

$$z^n = \sum_{k=0}^{\infty} \pi_{kn} \tilde{p}_k(z). \tag{15·4}$$

The *transposed inverse* of any basic set is defined to be the set $\{\hat{p}_n(z)\}$, where

$$\hat{p}_n(z) = \sum_{k=0}^{\infty} \pi_{kn} z^k, \tag{15·5}$$

$$z^n = \sum_{k=0}^{\infty} p_{kn} \hat{p}_k(z). \tag{15·6}$$

Given two basic sets $\{p_k^{(1)}(z)\}$, $\{p_k^{(2)}(z)\}$, we define their *product set* (in this order) to be the set $\{p_k(z)\}$, where

$$p_k(z) = \sum_{j=0}^{\infty} p_{kj}^{(1)} p_j^{(2)}(z), \tag{15·7}$$

$$z^n = \sum_{k=0}^{\infty} \pi_{nk} p_k(z), \tag{15·8}$$

$$\pi_{nk} = \sum_{i=0}^{\infty} \pi_{ni}^{(2)} \pi_{ik}^{(1)}. \tag{15·9}$$

All of these sets are, of course, only formally defined, and our first problem is to investigate their existence; for the inverse, for example, we need to show that $\check{p}_n(z)$ is in $H(\rho)$ or $\bar{H}(\rho)$ for some ρ and all n and that z^n is, for each n, represented by (15·2) in some $D(R)$, $D_+(R)$ or $\overline{D(R)}$.

Associated with any of these sets we have certain entities, such as the Cannon functions, for which we have introduced a standard notation. The set to which the entity refers will be indicated in the notation on the same system as above, e.g. $\check{\kappa}(r)$, $\tilde{\lambda}(R)$, $\hat{M}_k(\rho)$, $p_{kj}^{(1)}$, $\pi_{ni}^{(2)}$.

16. The transposed inverse

Let $\{p_k(z)\}$ be a basic set (on $\bar{H}(r)$), effective for $H(\rho)$ in $D_+(r)$. Then if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\rho), \quad \Pi_k f(0) = \sum_{n=0}^{\infty} a_n \pi_{nk}$$

exists and we saw in § 3 that $\sum_{k=0}^{\infty} \pi_{kn} z^k \in \bar{H}(1/\rho)$. This suggests that we define, more generally, the transposed inverse of a basic set $\{p_k\}$ on an $\mathscr{F}_+$-space F to be the set $\{\Pi_k\}$. Then if $\{p_k\}$ is effective for F_s in $\mathscr{T}_+$, $\{\Pi_k\}$ is a set of points in the dual F_s^* of F_s. Moreover, $\{p_k\}$ is effective for F_s in some $\mathscr{T}_\sigma^{(\omega)}$, so that, for all $x \in F_s$, $x = \sum_{k=0}^{\infty} \Pi_k(x)\, p_k \; (\mathscr{T}_\sigma^{(\omega)})$. Hence if $\Xi \in F_\sigma^*$,

$$\Xi(x) = \sum_{k=0}^{\infty} \Pi_k(x)\, \Xi(p_k).$$

Now the restriction of Ξ to F_s is continuous in $\mathscr{T}_s^{(\omega)}$, so that we may regard F_σ^* as a subspace of F_s^*, and conclude that each Ξ is represented in the weak topology on F_s^* by the series $\sum_{k=0}^{\infty} \Xi(p_k)\, \Pi_k$. This shows in particular that $\{\Pi_k\}$ is actually basic on F_s^*, since $\mathrm{Z}_n = \sum_{k=0}^{\infty} \mathrm{Z}_n(p_k)\, \Pi_k$, convergent in the weak topology on F_s^*. Moreover, since $p_k = \sum_{n=0}^{\infty} \mathrm{Z}_n(p_k)\, z_n \; (\mathscr{T}_\sigma^{(\omega)})$ and $\Xi \in F_\sigma^*$, we have $\Xi(p_k) = \sum_{n=0}^{\infty} \mathrm{Z}_n(p_k)\, \Xi(z_n)$, showing that the series $\sum_{k=0}^{\infty} \Xi(p_k)\, \Pi_k$ are basic. Summarizing:

Theorem 16·1. *If $\{p_k\}$ is effective for F_s in $\mathscr{T}_+$, then its transposed inverse $\{\Pi_k\}$ exists and is effective for some F_σ^* in the weak topology on F_s^*.*

Let us now suppose that $\{z_n\}$ is an absolute base for F_s satisfying the condition (2·4), i.e. that for each i, $\sum_{n=0}^{\infty} |z_n|_s^{(i)}\, |\mathrm{Z}_n|_s^{(i+1)} < \infty$; suppose also that $\{p_k\}$ is absolutely effective for F_s in $\mathscr{T}_\sigma^{(\omega)}$. Then each $\Xi \in F_\sigma^*$ is continuous in some $\mathscr{T}_\sigma^{(\iota)}$; and for some M and i we have $\bar{q}_\sigma^{(\iota)}(z_n) \leqslant M\, |z_n|_s^{(i)}$. Hence

$$\begin{aligned}\sum_{k=0}^{\infty} |\Xi(p_k)|\, |\Pi_k|_s^{(i+1)} &\leqslant |\Xi|_\sigma^{(\iota)} \sum_{k=0}^{\infty} |p_k|_\sigma^{(\iota)}\, |\Pi_k|_s^{(i+1)} \\ &\leqslant |\Xi|_\sigma^{(\iota)} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} |p_k|_\sigma^{(\iota)}\, |\Pi_k(z_n)|\, |\mathrm{Z}_n|_s^{(i+1)} \\ &= |\Xi|_\sigma^{(\iota)} \sum_{n=0}^{\infty} \bar{q}_\sigma^{(\iota)}(z_n)\, |\mathrm{Z}_n|_s^{(i+1)} \\ &\leqslant M\, |\Xi|_\sigma^{(\iota)} \sum_{n=0}^{\infty} |z_n|_s^{(i)}\, |\mathrm{Z}_n|_s^{(i+1)} < \infty.\end{aligned}$$

We have now proved:

THEOREM 16·2. *If F_s has an absolute base $\{z_n\}$ satisfying (2·4), and if $\{p_k\}$ is absolutely effective for F_s in $\mathscr{T}_\sigma^{(\omega)}$, then $\{\Pi_k\}$ is absolutely effective for F_σ^* in any $\mathscr{F}_+$-topology on F_s^*.*

Returning now to the case where $F = \bar{H}(r)$, $F_s = H(\rho)$, we can state immediately:

THEOREM 16·3. *If $\{p_k(z)\}$ is effective or absolutely effective for $H(\rho)$ in $D_+(r)$, then its transposed inverse exists and is effective or absolutely effective for $H(1/r)$ in $D_+(1/\rho)$.*

THEOREM 16·4. *Effectiveness of $\{p_k(z)\}$ in $D(R)$ is equivalent to effectiveness of $\{\hat{p}_k(z)\}$ in $D_+(1/R)$; similarly, effectiveness of $\{p_k(z)\}$ in $D_+(R)$ is equivalent to effectiveness of $\{\hat{p}_k(z)\}$ in $D(1/R)$. Corresponding results hold for absolute effectiveness.*

There is no such reciprocity for representation in $\overline{D(R)}$. For example, the W-basic set $p_0(z) = 1$, $p_n(z) = z^n - z^{n-1}$ is effective in $\overline{D(1)}$, whereas this is impossible for its transposed inverse $\hat{p}_n(z) = z^n/(1-z)$, since these functions are not in $\bar{H}(1)$.

Obviously, a transposed inverse will be U-basic if and only if the original set is U-basic.

The results of this section enable us to give a generalization of the strong form of Cauchy's inequality (cf. the inequality (12·5)).

First we define the *conjugate* of a basic set to be the set whose matrices are the complex conjugates of those of the original set. It is trivial that *every set is equivalent* (as regards effectiveness) *to its conjugate set.*

Now let us take, for simplicity, $R = 1$, and suppose that the U-basic set $\{p_k(z)\}$ is effective in $D(1)$ and in $\overline{D(1)}$ (these restrictions could be weakened). Then $\{\hat{p}_k(z)\}$ is effective in $D_+(1)$, hence so is its conjugate set $\{q_k(z)\}$. Thus if $f \in \bar{H}(1)$, we have, in $\overline{D(1)}$,

$$f(z) = \sum_{n=0}^{\infty} \alpha_n p_n(z) = \sum_{k=0}^{\infty} \beta_k q_k(z).$$

Denoting complex conjugates by an asterisk, we shall show that

$$\sum_{n=0}^{\infty} \alpha_n \beta_n^* = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|^2 \, d\theta, \tag{16·1}$$

which is the generalization referred to above. If the matrix (p_{nk}) is unitary, $\alpha_n = \beta_n$ and the equation reduces to

$$\sum_{n=0}^{\infty} |\alpha_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|^2 \, d\theta. \tag{16·2}$$

The inequality $\sum_{n=0}^{\infty} \alpha_n \beta_n^* \leqslant \max_{|z|=1} |f(z)|^2$ reduces in the case of Taylor series to Cauchy's inequality.

To prove (16·1), we first observe that on the unit circle γ, $\hat{p}_k(z^*) = \hat{p}_k(1/z) = z\pi_k(z)$, so that

$$f^*(z) = \sum_{k=0}^{\infty} \beta_k^* q_k^*(z) = \sum_{k=0}^{\infty} \beta_k^* \hat{p}_k(z^*) = \sum_{k=0}^{\infty} \beta_k^* z\pi_k(z),$$

and $$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|^2 \, d\theta = \frac{1}{2\pi i}\int_\gamma f(z) f^*(z) \, z^{-1} \, dz = \sum_{n=0}^{\infty} \alpha_n \sum_{k=0}^{\infty} \beta_k^* \frac{1}{2\pi i}\int_\gamma p_n(z) \, \pi_k(z) \, dz,$$

the term-by-term integration being validated by uniform convergence on γ.

17. The normalization condition

Contrary to what happens in the case of the transposed inverse, it is not, in general, true that effectiveness of two given sets implies effectiveness of their product set; or that effectiveness of a given U-basic set implies that its inverse is also effective (see, for example, Nassif 1951). We do, however, obtain such results if we place restrictions on the sets involved. We define

$$\mu(r) = \overline{\lim_{k\to\infty}}\,[M_k(r)]^{1/k}, \tag{17·1}$$

$$\nu(r) = \underline{\lim_{k\to\infty}}\,[M_k(r)]^{1/k}. \tag{17·2}$$

These are obviously increasing functions of r, so that left and right limits exist. We shall say that $\{p_k(z)\}$ *is normalized in* $D_+(r)$ *with constants* a, b (or briefly $N(a,b)$ in $D_+(r)$) if there are strictly positive constants a, b such that

$$\mu(r+)\leqslant br, \qquad \text{N (i)}$$

$$\nu(\rho)>ar \quad \text{for} \quad \rho>r. \qquad \text{N (ii)}$$

We shall also say that $\{p_k(z)\}$ *is normalized in* $D(R)$ *with constants* a, b (or briefly $N(a,b)$ in $D(R)$), if, for strictly positive a, b,

$$\mu(r)<bR \quad \text{for} \quad r<R, \qquad \text{(i)}$$

$$\nu(R-)\geqslant aR. \qquad \text{(ii)}$$

The cases $r=0$, $R=\infty$ are, as usual, included; a set is normalized at the origin if $\mu(0+)=0$ and $\nu(\rho)>0$ for $\rho>0$, and in $D(\infty)$ if $\nu(\infty)=\infty$ and $\mu(r)$ is finite for all finite r (the constants have now disappeared).

In what follows we shall suppose that $\{p_k(z)\}$ is a U-basic set, $N(a,b)$ and effective in $D_+(r)$. In the notation of §12 we have $z\hat{p}_k(z)=\pi_k(1/z)$, so that by theorem 12·2

$$\mu(R)\,\hat{\nu}(1/\rho)\leqslant 1$$

for $\rho>r$ and R such that $\kappa(R)<\rho$. Also, equation (12·5) gives $\mu(\rho)\,\hat{\nu}(1/\rho)\geqslant 1$, and we find easily

$$\mu(r+)\,\hat{\nu}(r^{-1}-)=1; \tag{17·3}$$

and since we may interchange μ and ν,

$$\nu(r+)\,\hat{\mu}(r^{-1}-)=1. \tag{17·4}$$

We see that N (i) is equivalent to $\quad \hat{\nu}(r^{-1}-)\geqslant 1/br, \qquad (17·5)$
and that N (ii) is equivalent to

$$\hat{\mu}(1/\rho)<1/ar \quad \text{for all} \quad \rho>r. \tag{17·6}$$

With theorem 16·4, the last two equations give:

Theorem 17·1. *If a U-basic set is* $N(a,b)$ *and effective in* $D_+(r)$ *then its transposed inverse is* $N(b^{-1},a^{-1})$ *and effective in* $D(r^{-1})$, *and conversely.*

Given $\rho>r$, choose R such that $r<R<1/a\hat{\mu}(1/\rho)$. Then

$$\check{M}_n(aR)\leqslant \sum_{k=0}^{\infty}|\pi_{nk}|\,(aR)^k\leqslant \rho^n\sum_{k=0}^{\infty}(aR)^k\,\hat{M}_k(1/\rho)<\infty,$$

by Cauchy's inequality, since $\hat{p}_k(z) \in H(1/r)$ (by theorem 16·4). This shows that $\check{p}_k(z) \in \bar{H}(ar)$ and that

$$\check{\mu}(ar+) \leqslant r. \tag{17·7}$$

Similarly, if $R > r$, then $bR > \mu(r+)$, and we can find $\rho > r$ such that $\mu(\rho) < bR$. For such R, ρ we have

$$\tilde{M}_k(1/bR) \leqslant \sum_{n=0}^{\infty} |p_{nk}| (bR)^{-n} \leqslant \rho^{-k} \sum_{n=0}^{\infty} (bR)^{-n} M_n(\rho) < \infty,$$

showing that $\tilde{p}_k(z) \in H(1/br)$ and that

$$\tilde{\mu}(1/bR) < 1/r \quad \text{for} \quad R > r. \tag{17·8}$$

Let us now suppose that $a = b$ and that $R > r$ is such that $\check{p}_n(z) \in \bar{H}(bR)$. Then using Schwarz's and Cauchy's inequalities,

$$\sum_{k=0}^{\infty} |\pi_{nk} p_{km}| \leqslant \left[\sum_{k=0}^{\infty} |\pi_{nk}|^2 (bR)^{2k}\right]^{1/2} \left[\sum_{k=0}^{\infty} |p_{km}|^2 (bR)^{-2k}\right]^{1/2} \leqslant \check{M}_n(bR)\, \tilde{M}_m(1/bR).$$

If we now choose $\rho > r$ so that $\rho\tilde{\mu}(1/bR) < 1$, we have

$$\omega_n(\rho) \leqslant \sum_{k=0}^{\infty} |\pi_{nk}| \sum_{m=0}^{\infty} |p_{km}| \rho^m \leqslant \check{M}_n(bR) \sum_{m=0}^{\infty} \rho^m \tilde{M}_m(1/bR) < \infty,$$

and $\lambda(r+) \leqslant \lambda(\rho) \leqslant \check{\mu}(bR)$. This being true for $R > r$, we have $\lambda(r+) \leqslant r$ by (17·7). We have proved:

THEOREM 17·2. *If a U-basic set is $N(b, b)$ and effective in $D_+(r)$, then it is absolutely effective in $D_+(r)$.*

This result indicates the strength of our normalization hypothesis. Similarly, for open circles, we have:

THEOREM 17·3. *If a U-basic set is $N(b, b)$ and effective in $D(R)$, then it is absolutely effective in $D(R)$.*

For by theorem 17·1, the hypotheses imply that the transposed inverse is $N(b^{-1}, b^{-1})$ and effective in $D_+(1/R)$, hence absolutely effective there by theorem 17·2. The result now follows from theorem 16·4.

18. The inverse of a U-basic set

We again suppose that $\{p_k(z)\}$ is U-basic, $N(a, b)$ and effective in $D_+(r)$. Then for any $\rho > r$, we can find R such that $r < R < 1/a\hat{\mu}(1/\rho)$. Hence, using Schwarz and Cauchy as in the previous section,

$$\check{\omega}_n(aR) = \sum_{k=0}^{\infty} |p_{nk}| \check{M}_k(aR) \leqslant \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} |p_{nk}\pi_{km}| (aR)^m \leqslant M_n(\rho) \sum_{m=0}^{\infty} (aR)^m \hat{M}_m(1/\rho) < \infty,$$

and the absolute convergence shows that the condition (15·2) for the inverse to be a basic set holds in $D_+(ar)$. Moreover, $\check{\lambda}(ar+) \leqslant \check{\lambda}(aR) \leqslant \mu(\rho)$ for any $\rho > r$, and by N (i),

$$\check{\lambda}(ar+) \leqslant br. \tag{18·1}$$

In particular:

THEOREM 18·1. *If a U-basic set is $N(b, b)$ and effective in $D_+(r)$, then its inverse is U-basic and absolutely effective in $D_+(br)$.*

It will be seen from (17·7) and (17·8) that the inverse is also $N(b^{-1}, b^{-1})$ in $D_+(br)$. Correspondingly, for open circles:

THEOREM 18·2. *If a U-basic set is $N(b, b)$ and effective in $D(R)$, then its inverse is U-basic and absolutely effective in $D(bR)$.*

For the hypotheses show that the transposed inverse $\{\hat{p}_k(z)\}$ is $N(b^{-1}, b^{-1})$ and effective in $D_+(1/R)$, so that by the previous theorem $\{\tilde{p}_k(z)\}$ is absolutely effective in $D_+(1/bR)$. Since $\{\breve{p}_k(z)\}$ is the transposed inverse of $\{\tilde{p}_k(z)\}$ the rest follows from theorem 16·4.

19. THE TRANSPOSE OF A U-BASIC SET

THEOREM 19·1. *If a U-basic set is $N(b, b)$ and effective in $D_+(r)$, then its transpose is U-basic and absolutely effective in $D(1/br)$.*

For by theorem 18·1, $\{\breve{p}_k(z)\}$ is absolutely effective in $D_+(br)$, and $\{\tilde{p}_k(z)\}$ is its transposed inverse. In theorem 18·2, we proved incidentally:

THEOREM 19·2. *If a U-basic set is $N(b, b)$ and effective in $D(R)$, then its transpose is U-basic and absolutely effective in $D_+(1/bR)$.*

20. PRODUCT SETS

THEOREM 20·1. *If $\{p_k^{(2)}(z)\}$ is U-basic, $N(b, b)$ and effective in $D_+(r)$, and if $\{p_k^{(1)}(z)\}$ is absolutely effective in $D_+(br)$, then their product set $\{p_k(z)\}$ is basic and absolutely effective in $D_+(r)$.*

Proof. Choose $\rho > br$ small enough for $\breve{M}_n^{(2)}(\rho) < \infty$ (cf. (17·7)). By hypothesis $\lambda^{(1)}(br+) = br$, so that we can find $R > r$ such that $\lambda^{(1)}(bR) < \rho$. By (17·8) we can choose r_1 such that $\tilde{\mu}^{(2)}(1/bR) < 1/r_1 < 1/r$. Hence, by the usual use of Schwarz's and Cauchy's inequalities,

$$\omega_n(r_1) \leqslant \sum_{i,j,k,l} |\pi_{ni}^{(2)} \pi_{ik}^{(1)} p_{kj}^{(1)} p_{jl}^{(2)}| r_1^l \leqslant \breve{M}_n^{(2)}(\rho) \sum_{k=0}^{\infty} M_k^{(1)}(bR)\, \hat{M}_k^{(1)}(1/\rho) \sum_{l=0}^{\infty} r_1^l \tilde{M}_l^{(2)}(1/bR),$$

which is finite by the choice of r_1, R and ρ and by theorem 12·5. The absolute convergence shows that $p_k(z) \in \bar{H}(r)$ and that the condition (15·8) for $\{p_k(z)\}$ to be basic holds in $D_+(r)$. Moreover, $\lambda(r+) \leqslant \lambda(r_1) \leqslant \breve{\mu}^{(2)}(\rho)$ for $\rho > br$, and the rest follows from (17·7).

THEOREM 20·2. *If $\{p_k^{(2)}(z)\}$ is U-basic, $N(b, b)$ and effective in $D(R)$, and if $\{p_k^{(1)}(z)\}$ is absolutely effective in $D(bR)$, then their product set $\{p_k(z)\}$ is basic and absolutely effective in $D(R)$.*

Proof. The hypotheses imply that $\{\hat{p}_n^{(2)}(z)\}$ is U-basic, $N(b^{-1}, b^{-1})$ and effective in $D_+(1/R)$ and that $\{\hat{p}_n^{(1)}(z)\}$ is absolutely effective in $D_+(1/bR)$. By the previous theorem, their product set exists and is absolutely effective in $D_+(1/R)$, and its transposed inverse is the product set $\{p_k(z)\}$, since obviously $p_n(z) = \sum_{k=0}^{\infty} p_{nk} z^k$, where $p_{nk} = \sum_{j=0}^{\infty} p_{nj}^{(1)} p_{jk}^{(2)}$.

With the same hypotheses on $\{p_k^{(2)}(z)\}$ as in theorem 20·1, let us suppose only that $\{p_k^{(1)}(z)\}$ is effective in $D_+(br)$. Relative to the absolute base $\{p_k^{(2)}(z)\}$, the set $\{p_k(z)\}$ will be basic if $p_k(z) \in \bar{H}(r)$ and if $p_n^{(2)}(z)$ is represented in $D_+(r)$ by the series $\sum_{k=0}^{\infty} \pi_{nk}^{(1)} p_k(z)$; it will follow *a posteriori* that $\{p_k(z)\}$ is basic under (15·8), by the remark at the end of § 9.

For $\{p_k(z)\}$ to be effective in $D_+(r)$, it is necessary and sufficient, by theorem 7·3, that given $\rho > r$, we can find $R > r$ and M such that $F_n^*(R) = \max_{\mu,\nu} \max_{|z|=R} \left| \sum_{k=\mu}^{\nu} \pi_{nk}^{(1)} p_k(z) \right| \leqslant M M_n^{(2)}(\rho)$;

hence sufficient that $\kappa^*(r+) < \nu^{(2)}(\rho)$ for all $\rho > r$. From N (ii), we see that it is sufficient to prove that $\kappa^*(r+) \leqslant br$.

Given $\rho > br$, we can find $R > r$ such that $\mu^{(2)}(R) < \rho$ (by N (i)). Hence, using Cauchy's inequality,

$$F_n^*(R) = \max_{\mu,\nu} \max_{|z|=R} \left| \sum_{j=0}^{\infty} p_j^{(2)}(z) \sum_{k=\mu}^{\nu} \pi_{nk}^{(1)} p_{kj}^{(1)} \right|$$

$$\leqslant \sum_{j=0}^{\infty} M_j^{(2)}(R) \max_{\mu,\nu} \left| \sum_{k=\mu}^{\nu} \pi_{nk}^{(1)} p_{kj}^{(1)} \right|$$

$$\leqslant \sum_{j=0}^{\infty} M_j^{(2)}(R)\, \rho^{-j} F_n^{(1)}(\rho) < \infty.$$

From the uniform convergence in μ and ν and the convergence of $\sum_{k=0}^{\infty} \pi_{nk}^{(1)} p_{kj}^{(1)}$, we see that $\{p_k(z)\}$ is basic, with respect to the base $\{p_n^{(2)}(z)\}$, on $\bar{H}(r)$. Moreover,

$$\kappa^*(r+) \leqslant \kappa^*(R) \leqslant \kappa^{(1)}(\rho) \to \kappa^{(1)}(br+) = br.$$

We have proved:

Theorem 20·3. *If the U-basic set $\{p_k^{(2)}(z)\}$ is $N(b,b)$ and effective in $D_+(r)$, and if $\{p_k^{(1)}(z)\}$ is effective in $D_+(br)$, then their product set is basic and effective in $D_+(r)$.*

By the same argument as in theorem 20·2, we have also:

Theorem 20·4. *If the U-basic set $\{p_k^{(2)}(z)\}$ is $N(b,b)$ and effective in $D(R)$, and if $\{p_k^{(1)}(z)\}$ is effective in $D(bR)$, then their product set is basic and effective in $D(R)$.*

Finally, we observe that if, in any of these theorems, both $\{p_k^{(1)}(z)\}$ and $\{p_k^{(2)}(z)\}$ are U-basic, then so is their product $\{p_k(z)\}$. For by § 18, $p_k^{(1)}(z) = \sum_{l=0}^{\infty} p_{kl} \check{p}_l^{(2)}(z)$, and by theorem 6·1

$$\delta_k^n = \sum_{l=0}^{\infty} p_{kl} \sum_{i=0}^{\infty} \pi_{li}^{(2)} \pi_{in}^{(1)} = \sum_{l=0}^{\infty} p_{kl} \pi_{ln}.$$

21. Special sets of polynomials

Let $\{p_k(z)\}$ be W-basic, and denote by D_n the degree of the polynomial of highest degree appearing in the representation

$$z^n = \sum_k \pi_{nk} p_k(z).$$

We find, easily, $\quad D_n \geqslant n \quad$ and $\quad N_n \leqslant D_n + 1.$

Hence any of the following hypotheses imply that $\{p_k(z)\}$ is C-basic:

$$\lim_{n\to\infty} D_n^{1/n} = 1, \tag{21·1}$$

$$D_n = O(n), \tag{21·2}$$

$$\lim_{n\to\infty} D_n/n = 1, \tag{21·3}$$

$$p_n(z) \text{ is of degree } n. \tag{21·4}$$

Any of these properties imply those preceding. A set satisfying (21·4) is called *simple*.

For the second hypothesis, we have (Whittaker 1949, T_{10}):

Theorem 21·1. *If $\{p_n(z)\}$ is such that $D_n = O(n)$ and $\lambda(0+) < \infty$, then $\lambda(r)$ is continuous in $0 < r < \infty$. We have, moreover, for $0 < r \leqslant R < \infty$, $r^c\lambda(R) \leqslant R^c\lambda(r)$, where $c = \overline{\lim_{n\to\infty}}\,(D_n/n)$.*

This gives the immediate:

Corollary. For a set such that $D_n = O(n)$ *effectiveness in* $\overline{D(R)}$ *and in* $D_+(R)$ *are equivalent; also effectiveness in* $D(R)$ *implies effectiveness in* $\overline{D(R)}$.

For the third hypothesis, we have (Whittaker 1949, T_{12}):

THEOREM 21·2. *If a basic set for which* $D_n/n \to 1$ *is effective in* $D(r)$, $\overline{D(r)}$ *or in* $D_+(r)$, *then it is effective in* $D(R)$, $\overline{D(R)}$ *and in* $D_+(R)$ *for all* $R>r$.

Corollary. If a basic set for which $\hat{D}_n/n \to 1$ (i.e. the transposed inverse of which satisfies (21·3)) *is effective in* $D(R)$, $\overline{D(R)}$ *or in* $D_+(R)$, *then it is effective in* $D(r)$, $\overline{D(r)}$ *and* $D_+(r)$ *for all* $r<R$; *effectiveness in* $\overline{D(R)}$ *implies effectiveness in* $D(R)$.

This is a simple consequence of the theorem and of theorem 16·3, and applies in particular to transposes of simple sets.

For simple sets we have, in addition to the above results, a convenient criterion for normalization. Suppose

$$0 < a = \varliminf_{n\to\infty} |p_{nn}|^{1/n} \leqslant \varlimsup_{n\to\infty} |p_{nn}|^{1/n} = b < \infty. \tag{21·5}$$

Then since $p_{nn}\pi_{nn} = 1$, by Cauchy's inequality,

$$|p_{nn}| R^n \leqslant M_n(R) = p_{nn}\pi_{nn} M_n(R) \leqslant |p_{nn}| F_n(R),$$

and $aR \leqslant \nu(R) \leqslant \mu(R) \leqslant b\kappa(R)$. Hence, *a simple set satisfying* (21·5) *is* $N(a,b)$ *in* $D(R)$ *or in* $D_+(r)$ *whenever it is effective there*, and when $a=b$, the results of the preceding sections apply.

In particular, a simple set with leading coefficient unity will be $N(1,1)$ whenever it is effective, so that our results include those of Eweida (Whittaker 1949, T_{38} and T_{39}). Theorems 20·3 and 18·1 also include the results of Nassif (1951), where the hypothesis used for effectiveness at the origin is $0 < aR \leqslant \nu(R) \leqslant \mu(R) \leqslant bR < \infty$ for all R in $0<R<\epsilon$. This hypothesis is rather strong, so that Nassif's result does not imply that of Eweida (consider, for example, Whittaker 1949, p. 21, E_9). Our results include also those of Tantaoui (1950).

22. Representation in closed circles

In the previous sections we have been concerned exclusively with representation in $D_+(r)$ or in $D(R)$. If, in the results of §§ 18 to 20, we impose the extra condition that $D_n = O(n)$, then effectiveness in $D_+(R)$ and in $\overline{D(R)}$ are equivalent (theorem 21·1, corollary), and the results can be stated in terms of effectiveness in $\overline{D(R)}$.

These remarks apply in particular to simple sets with leading coefficient unity, so that our results include also those of Nassif and of Mursi & Makar (Whittaker 1949, T_{35} and T_{37}). It must be pointed out that the corresponding results for $D(R)$, though true (by the preceding), do not follow from those for $\overline{D(R)}$, as stated by Professor Whittaker; for the converse of theorem 21·1, corollary (Whittaker 1949, T_{36}) is false.

In our results on effectiveness in $\overline{D(R)}$, we are, of course, using the hypothesis that the set is normalized in $D_+(R)$, and the question arises as to whether a weaker hypothesis would suffice for this special case. An attempt in this direction has been made by Nassif (1947), who uses the hypothesis

$$0 < \lim_{n\to\infty} [M_n^{(2)}(R)]^{1/n} = aR < \infty, \tag{22·1}$$

or, in our notation, $0 < \mu^{(2)}(R) = \nu^{(2)}(R) = aR < \infty$. His assertions are:

(A) *Let* $\{p_n^{(1)}(z)\}$ *be C-basic and* $\{p_n^{(2)}(z)\}$ *such that* $D_n^{(2)} = O(n)$. *Suppose also that* $\{p_n^{(2)}(z)\}$ *is effective in* $\overline{D(R)}$. *Then for the product set* $\{p_k(z)\}$ *to be effective in* $\overline{D(R)}$ *it is necessary and sufficient that*

$$\overline{\lim_{n\to\infty}}\,[\psi_n(R)/M_n^{(2)}(R)]^{1/n} = 1,$$

where
$$\psi_n(R) = \sum_{k=0}^{\infty} |\pi_{nk}^{(1)}|\, M_k(R).$$

(B) *Let* $\{p_n^{(1)}(z)\}$, $\{p_n^{(2)}(z)\}$ *be as in* (A). *Suppose also that* $\{p_n^{(2)}(z)\}$ *satisfies* (22·1) *and that* $\lambda^{(1)}(r)$ *is continuous at* $r = aR$. *Then the product set* $\{p_n(z)\}$ *will be effective in* $\overline{D(R)}$ *if and only if* $\{p_n^{(1)}(z)\}$ *is effective in* $\overline{D(aR)}$.

We shall show by examples that the assertion (A) is false, the condition being neither necessary nor sufficient, and one of these examples will also show that the conditions in (B) are insufficient for the effectiveness of the product set. The rest of (B) and the rest of the paper, all of which depend for their proof on (A), are accordingly unproved and will require re-examination.

The fault in Nassif's proof of 'necessity' in (A) lies in the assumption that if two sets $\{p_k(z)\}$, $\{q_k(z)\}$ are both effective in $\overline{D(R)}$ and if $f(z)$ is represented (basically) in $\overline{D(R)}$ by one set, then $f(z)$ will be represented (basically) in $\overline{D(R)}$ by the other; for an $f(z)$ regular in $D(R)$ and continuous (but not regular) in $\overline{D(R)}$, this implication is not obvious *a priori*, and turns out to be false in general. In the proof of 'sufficiency' there is confusion between the number D_n and the degree of $p_n(z)$.

Example 1. Consider the simple set $\{p_n^{(1)}(z)\}$, given by

$$p_n^{(1)}(z) = \sum_{k=0}^{n} z^k, \quad \pi_{nk}^{(1)} = \delta_k^n - \delta_k^{n-1}.$$

It is easy to verify that $\lambda^{(1)}(r) = 1$ if $r \leqslant 1$, $\lambda^{(1)}(r) = r$ if $r > 1$. In particular, $\{p_n^{(1)}(z)\}$ *is C-basic, effective in* $\overline{D(1)}$, *and such that* $\lambda^{(1)}(r)$ *is continuous at* $r = 1$.

Consider also the set $\{p_k^{(2)}(z)\}$, given by

$$p_k^{(2)}(z) = \begin{cases} z^{2k} & \text{if } k \text{ is an odd prime,} \\ z^m & \text{if } k = 2^m \text{ and } m \text{ is an odd prime,} \\ z^k & \text{otherwise.} \end{cases}$$

This being simply a rearrangement of the Taylor base, it is everywhere effective; $M_k^{(2)}(1) = 1$ for all k; moreover, since $z^n = p_k^{(2)}(z)$ for some k, we have $N_n^{(2)} = 1$ and $D_n^{(2)} = n$ for all n. Hence $\{p_k^{(2)}(z)\}$ *is such that* $D_n^{(2)} = O(n)$, *effective in* $\overline{D(1)}$, *and satisfies* (22·1) *at* $R = 1$ *with* $a = 1$. Forming the product set $\{p_k(z)\}$, we have $p_n(z) = \sum_{k=0}^{n} p_k^{(2)}(z)$. Also, since obviously $\{p_k^{(2)}(z)\}$ is its own inverse,

$$\pi_{nk}^{(2)} = \begin{cases} \delta_k^{2n} & \text{if } n \text{ is an odd prime,} \\ \delta_k^m & \text{if } n = 2^m \text{ and } m \text{ is an odd prime,} \\ \delta_k^n & \text{otherwise.} \end{cases}$$

Hence $\pi_{nk} = \sum_{l=0}^{\infty} \pi_{nl}^{(2)} \pi_{lk}^{(1)} = \pi_{nk}^{(2)} - \pi_{n,k+1}^{(2)}$, $N_n = 2$ and $\{p_n(z)\}$ *is C-basic*. Also $M_n(1) = n+1$ or all n, so that

$$\psi_n(1) = \sum_{k=0}^{\infty} |\pi_{nk}^{(1)}| M_k(1) = M_n(1) + M_{n-1}(1) = 2(n+1).$$

Hence $\overline{\lim_{n\to\infty}} [\psi_n(1)/M_n^{(2)}(1)]^{1/n} = 1$ so that all the requirements for 'sufficiency' in (A) and (B) are satisfied. However, the product set $\{p_k(z)\}$ is not effective in $\overline{D(1)}$. For if n is an odd prime,

$$\omega_n(1) \geqslant F_n(1) \geqslant |\pi_{n2^n}| M_{2^n}(1) > 2^n \quad \text{and} \quad \lambda(1) \geqslant 2.$$

Example 2. Take $\{p_n^{*(1)}(z)\}$ to be the product set $\{p_n(z)\}$ of the previous example; take $\{p_n^{*(2)}(z)\}$ to be the same inner set $\{p_n^{(2)}(z)\}$ as in the previous example. Then their product set $\{p_n^*(z)\}$ is the outer set $\{p_n^{(1)}(z)\}$ of the previous example (since $\{p_n^{(2)}(z)\}$ is its own inverse). We saw above that $\{p_n^{*(1)}(z)\}$ is C-basic, $\{p_n^{*(2)}(z)\}$ effective in $\overline{D(1)}$ and such that $D_n^{(2)} = O(n)$, and $\{p_n^*(z)\}$ effective in $\overline{D(1)}$. Thus the requirements for 'necessity' in (A) are fulfilled, whilst, if n is an odd prime,

$$\psi_n^*(1) = \sum_{k=0}^{\infty} |\pi_{nk}^{*(1)}| M_k^*(1) = \sum_{k=0}^{\infty} |\pi_{nk}| M_k^{(1)}(1) \geqslant |\pi_{n2^n}| M_{2^n}^{(1)}(1) \geqslant 2^n,$$

$M_n^{*(2)}(1) = 1$ for all n, and therefore $\overline{\lim_{n\to\infty}} [\psi_n^*(1)/M_n^{*(2)}(1)]^{1/n} \geqslant 2$.

Let us write d_n for the degree of $p_n(z)$. Then if we write $\check{d}_n^{(2)}$ instead of $D_n^{(2)}$ in (A) and (B), Nassif's proof of sufficiency is then valid and we have the following result:

THEOREM 22·1. *Let* $\{p_k^{(2)}(z)\}$ *be such that* $\check{d}_n^{(2)} = O(n)$ *and effective in* $\overline{D(R)}$, *where it satisfies* (22·1); *let* $\{p_k^{(1)}(z)\}$ *be C-basic and effective in* $D_+(aR)$. *Then the product set is absolutely effective in* $\overline{D(R)}$.

The conditions of this theorem do not imply that $\{p_k^{(2)}(z)\}$ is either normalized or effective in $D_+(R)$, so that the theorem is not a corollary of our preceding results. Although the hypothesis $\check{d}_n = O(n)$ implies that the set is C-basic, it does not imply the corresponding relation for D_n or d_n (we have seen above, in example 1, that the hypothesis $D_n = O(n)$ does not imply the corresponding relation for d_n or $\check{d}_n$). These remarks are illustrated by the following example.

Example 3. Let $\phi(n)$ denote the number of odd primes less than n. Define $\{p_n(z)\}$ from the Taylor base $\{z^n\}$ by replacing z^n by $z^n + z^{2^n}$ whenever n is an odd prime, and displacing the corresponding z^{2^n} to the place immediately following. The first few are: $1, z, z^2, z^3 + z^8, z^8, z^4, z^5 + z^{32}, z^{32}, z^6, \ldots$. Now if n is not of the form 2^k where k is prime, the place of z^n in this list is increased by at most $\phi(n)$. Hence if n is not prime, $z^n = p_k(z)$, and if n is prime, $z^n = p_k(z) - p_{k+1}(z)$, where in either case $k \leqslant n + \phi(n)$. Thus $\check{d}_n \leqslant n + \phi(n) + 1$ and $\check{d}_n/n \to 1$. However, for some $k \leqslant n + \phi(n)$ with n prime, $d_k = 2^n$ and $D_n = 2^n$, so that neither is $O(n)$.

Since $M_n(1) \leqslant 2$ we have $\omega_n(1) \leqslant 3$, and the set satisfies (22·1) for $R = 1$ and is effective in $\overline{D(1)}$; it is easy to see that $\lambda(R) = \infty$ for $R > 1$ and that the set is not $N(1,1)$ in $D_+(1)$.

We point out also that the inverse of a C-basic set or the product of two C-basic sets need not be C-basic.

23. The integrated set and the derived set

Given a basic set $\{p_k(z)\}$ we have

$$z^n = \sum_{k=0}^{\infty} \pi_{nk}\, p_k(z). \tag{23·1}$$

Integrating, we have

$$z^{n+1} = \sum_{k=0}^{\infty} (n+1)\, \pi_{nk} \int_0^z p_k(t)\, \mathrm{d}t = \sum_{k=1}^{\infty} \pi^*_{n+1,k}\, p^*_k(z),$$

where we have written $p^*_k(z) = \int_0^z p_{k-1}(t)\, \mathrm{d}t$, $\pi^*_{nk} = n\pi_{n-1,k-1}$ $(n>0,\ k>0)$. Adjoining the function $p^*_0(z) = 1$, and defining $\pi^*_{nk} = \delta^n_k$ if $n = 0$ or $k = 0$, the set $\{p^*_k(z)\}$ forms a basic set called the *integrated set associated with* $\{p_k(z)\}$. It is easy to see that the integrated set is U-basic if and only if the given set is U-basic.

Suppose the U-basic set $\{p_k(z)\}$ is effective in $D(R)$ (or in $D_+(r)$); then if $f(z)$ is in $H(R)$ (or $\bar{H}(r)$), so is its derivative $f'(z)$, and so $f'(z)$ is represented in $D(R)$ (or in $D_+(r)$) by its basic series $\sum_{k=0}^{\infty} \Pi_k f'(0)\, p_k(z)$. Integrating this, we have, in the same region,

$$f(z) = f(0)\, p^*_0(z) + \sum_{k=1}^{\infty} \Pi_{k-1} f'(0)\, p^*_k(z). \tag{23·2}$$

This expansion must be unique, since, if it were not, we should obtain, by differentiation, non-basic representations of $f'(z)$ relative to the effective U-basic set $\{p_k(z)\}$. It follows (§9) that they are basic and that the integrated set is also effective in $D(R)$ (or $D_+(r)$).

More generally, if any basic set $\{p_k(z)\}$ is effective for $H(\rho)$ or $\bar{H}(\rho)$ in $D(r)$, $\overline{D(r)}$ or $D_+(r)$, then for any f in the appropriate class we obtain the representation (23·2) in the appropriate topology. It is easy to show by direct calculation that the series (23·2) is the basic series of f relative to the set $\{p^*_k(z)\}$, so that we may state:

Theorem 23·1. *The integrated set is effective whenever the given set is effective.*

If we now differentiate (23·1), we obtain

$$nz^{n-1} = \sum_{k=0}^{\infty} \pi_{nk}\, p'_k(z). \tag{23·3}$$

In particular, $\sum_{k=0}^{\infty} \pi_{0k}\, p'_k(z) = 0$, and at least one $\pi_{0k} \neq 0$. If ν is the smallest k for which this is true, we define $p^*_n(z) = p'_n(z)$ if $n<\nu$, and $p^*_n(z) = p'_{n+1}(z)$ if $n \geqslant \nu$. Eliminating $p'_\nu(z)$ from the remaining equations of (23·3), we obtain

$$z^n = \sum_{k=0}^{\infty} \pi^*_{nk}\, p^*_k(z),$$

where

$$\pi^*_{nk} = \begin{cases} \pi_{n+1,k}/(n+1) & \text{if } k<\nu, \\ (\pi_{n+1,k+1}\pi_{0\nu} - \pi_{n+1,\nu}\pi_{0,k+1})/(n+1)\,\pi_{0\nu} & \text{if } k \geqslant \nu. \end{cases}$$

The basic set $\{p^*_k(z)\}$ so defined is called the *derived set of the given set*. It may be verified that it is U-basic if and only if the given set is U-basic.

Theorem 23·2. *If a basic set is effective for* $H(\rho)$ *or* $\bar{H}(\rho)$ *in* $D(R)$ *or in* $D_+(R)$, *then so is its derived set.*

Given any f in $H(\rho)$ or $\bar{H}(\rho)$, we can represent $F(z) = \int_0^z f(t)\,dt$ by its basic series and differentiate it. It then only remains to show that the series obtained (after eliminating $p'_\nu(z)$) is basic relative to the derived set.

This argument fails for representation in $\overline{D(R)}$, since we can conclude, after differentiation, only that f is represented in $D(R)$. That the corresponding result to theorem 23·2 for $\overline{D(R)}$ is false in general has been shown by Mursi & Makar (1948), who give a sufficient condition for its validity (the condition (21·1) above). Under this condition, the converse of theorem 23·1 also holds.

24. Generalized Laurent series

The general results of part II apply not only to spaces of functions regular in simply-connected domains, but also when the domains are more general, provided that a base is given for the space. The simplest doubly-connected domain is the annulus

$$A(r_1, r_2) = \{z : r_1 < |z| < r_2\}$$

bounded by two circles. For the space $H(A)$ of functions regular in A, endowed with the usual topology of uniform convergence in any interior closed region, we have the Laurent base $\{z^n : n = 0, \pm 1, \pm 2, \ldots\}$; and the theory could easily be adapted to the usual notation $\sum_{-\infty}^{\infty} Z_n(x)\, z_n$ instead of $\sum_{n=0}^{\infty} Z_n(x)\, z_n$.

However, Laurent's theorem not merely proves that the Laurent base is a base for $H(A)$, but it constructs the Laurent base out of the Taylor base for $H(r_2)$. If we take $p_n(z) = z^n$ we find that $\pi_n(z) = z^{-n-1}$, so that the Laurent base is obtained by defining, for $n = 0, 1, 2, \ldots$,

$$p_{-n-1}(z) = \pi_n(z). \tag{24·1}$$

It is natural to inquire whether this property holds more generally. Let C_1, C_2 be circles of radii R_1, R_2 with $r_1 < R_1 < R_2 < r_2$. Then if $f \in H(A)$ and $z \in A(R_1, R_2)$ we have, formally,

$$\begin{aligned} f(z) &= (2\pi i)^{-1} \int_{C_2 - C_1} f(w)\,(w-z)^{-1}\,dw \\ &= (2\pi i)^{-1} \int_{C_2} f(w) \sum_{k=0}^{\infty} p_k(z)\,\pi_k(w)\,dw + (2\pi i)^{-1} \int_{C_1} f(w) \sum_{k=0}^{\infty} p_k(w)\,\pi_k(z)\,dw \\ &= \sum_{-\infty}^{\infty} \Pi_k f(0)\, p_k(z), \end{aligned}$$

where we have written
$$\Pi_k f(0) = (2\pi i)^{-1} \int_{C_2} f(w)\,\pi_k(w)\,dw, \tag{24·2}$$

$$\Pi_{-k-1} f(0) = (2\pi i)^{-1} \int_{C_1} f(w)\,p_k(w)\,dw. \tag{24·3}$$

The series $\sum_{-\infty}^{\infty} \Pi_k f(0)\, p_k(z)$ may be called the *generalized Laurent series of* f.

Theorem 24·1. *In order that every f in $H(A)$ shall be represented in A by its generalized Laurent series, it is necessary and sufficient that $\{p_n(z)\}$ should be effective in $D_+(r_1)$ and in $D(r_2)$. If $0 < r_1 < r_2 < \infty$, an equivalent condition is that $\kappa(r) = r$ for $r_1 \leqslant r < r_2$.*

Proof. The equivalence of the two conditions is an immediate consequence of the lemma in § 11. The sufficiency is a consequence of theorem 12·4, which justifies the formal operations by which the series were obtained. For the necessity, let $f \in H(r_2)$; then $f \in H(A)$ and hence is represented in A by its generalized Laurent series. But from (24·3) and Cauchy's theorem, $\Pi_{-k-1} f(0) = 0$ if $k \geqslant 0$, and by equation (12·1) the series reduces to the basic series of f. Hence each $f \in H(r_2)$ is represented in A, and hence (by the maximum modulus principle) in $D(r_2)$, by its basic series, i.e. $\{p_n(z)\}$ is effective in $D(r_2)$. Now suppose $g(z) \in H(1/r_1)$; then $z^{-1}g(z^{-1}) = f(z)$ is in $H(A)$. Since evidently $\Pi_k f(0) = 0$ for $k \geqslant 0$, we have

$$f(z) = \sum_{k=0}^{\infty} \Pi_{-k-1} f(0)\, p_{-k-1}(z) = \sum_{k=0}^{\infty} \Pi_k g(0)\, \pi_k(z),$$

since $\hat{\pi}_k(z) = z^{-1} p_k(z^{-1})$. Since also $\hat{p}_k(z) = z^{-1}\pi_k(z^{-1})$,

$$g(w) = \sum_{k=0}^{\infty} \hat{\Pi}_k g(0)\, \hat{p}_k(w).$$

This series represents g in $A(r_2^{-1}, r_1^{-1})$ and hence in $D(1/r_1)$, so that the transposed inverse of $\{p_k(z)\}$ is effective in $D(1/r_1)$. By theorem 16·4, $\{p_k(z)\}$ is effective in $D_+(r_1)$.

If the set $\{p_k(z)\}$ is U-basic, the representations will be unique, since $\{p_k(z)\}$ cannot represent zero in $D(r_2)$ nor can $\{\hat{p}_k(z)\}$ in $D(1/r_1)$. Conversely, unique representations are necessarily generalized Laurent series. Finally, from theorem 11·4, we obtain:

THEOREM 24·2. *There is an annulus $A(a, b)$ such that $\{p_n(z)\}$ is effective in A and in any interior annulus but in no other, save possibly annuli with zero inner radius or infinite outer radius.*

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